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Social Security, Economic Growth, and Social Welfare in an Overlapping Generations Model with Aggregate and Idiosyncratic Shocks

Toshiki Tamai*

Abstract

This paper presents a study of the growth and welfare effects of four public pension systems under aggregate and idiosyncratic shocks. The equilibrium growth rate obtained under the pay-as-you-go pension system is lower than the growth rate achieved under the funded pension systems because the unfunded pension system hinders capital accumulation. However, pay-as-you-go with additional benefits for saving enhances capital accumulation by incentivizing people to save. Particularly, the equilibrium growth rate under the modified unfunded pension system exceeds that under the funded pension system if the degree of relative risk aversion is sufficiently small. With regard to social welfare, within the Rawlsian welfare function, pay-as-you-go without saving credit is superior to the fully funded system if people are highly risk-averse. By contrast, if they have low risk aversion, then pay-as-you-go with saving credit is preferable. Considering the Benthamite welfare function, these results hold if the low-income classes have thick population. This finding implies that the demographic structure of income classes is important to ascertain the optimal extent of social security.

Keywords: Public Pension; Risk; Economic Growth
JEL Classifications: H55; O41

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1. Introduction

Economically developed countries provide their own social security programs to safeguard incomes of elderly persons during old age in cases of insufficient savings. Social security programs in most such countries are based on pay-as-you-go system, whereas a few countries entirely or partly manage their social security systems by funded systems.[1] The retirement benefits in pay-as-you-go systems are financed by contributions levied from people from current working generations. Because they do not depend directly on pensioners' paid contributions, pay-as-you-go systems have intergenerational redistribution effects. By contrast, a fully funded system provides retirement benefits that are perfectly related to the pensioners' earnings and contributions. Therefore such a system has no intergenerational redistribution effects.

Against this background, social security has been studied as a core issue in the fields of public finance, public economics, and macroeconomics.[2] Specifically addressing matters of savings and old age income, earlier studies of the literature have demonstrated that social security impedes capital accumulation and induces early retirement (e.g., Feldstein, 1974, 1977; Kotlikoff, 1979). Findings show that a fully funded system is preferable over a pay-as-you-go system in a dynamically efficient economy, with the former being superior to the latter when considering economic growth. However, the importance of social security is based on the presence of uncertainty because it affects sharing risks and the optimality of allocation under various shocks and improves social welfare (e.g., Enders and Lapan, 1982; Gordon and Varian, 1988; Thøgersen, 1998; Demange and Laroque, 1999; Wagener, 2004; Gottardi and Kubler, 2011).[3]

Recent studies of the literature describing social security treat issues of idiosyncratic shocks and the heterogeneity of individuals to elucidate the redistributive effects of social security (e.g., Conesa and Krueger 1999; Harenberg and Ludwig, 2015, 2019; Bagchi, 2019).[4] Idiosyncratic shocks affect personal earnings and generate income differences. Therefore, it is necessary to analyze effects of social security not only on intergenerational inequality but also on intragenerational inequality to consider idiosyncratic shocks. In particular, one study related to ours is an elaborate research developed by Harenberg and Ludwig (2015). They show that pay-as-you-go social security can give partial insurance against idiosyncratic and aggregate risks when

[1] For example, the pension system in Singapore, Central Provident Fund (CPF), is a social security system funded both by employers' and employees' contributions. Superannuation in Australia and the premium pension in Sweden are funded parts of the earnings-related pension, although these two countries have pay-as-you-go pension systems which work as the intragenerational redistribution devices.

[2] Issues related to social security have been discussed widely, for example, intergenerational risk sharing (e.g., Smith, 1982; Bohn, 2001; Gollier, 2008), adverse selection (e.g., Abel, 1986), and optimal social security (e.g., Samuelson, 1975; Sheshinski and Weiss, 1981).

[3] Hauenschild (2002) developed a general equilibrium model with stochastic production and social security to examine the existence, uniqueness, and stability of stochastic equilibrium.

[4] Bagchi (2019) specifically examines the presence of differential mortality. Regarding this, Kelly (2021) demonstrates that the assumption of mortality homogeneity biases the equilibrium growth rate and welfare analysis. Harenberg and Ludwig (2015, 2019) treat the aggregate and idiosyncratic earning shocks.

markets are incomplete. Their studies clarify insurance against risks and crowding-out of capital because of distortionary tax as key determinants of the welfare effects of social security.

In reality, social security programs are specifically operated depending on each country's economic circumstances, even though most are based on pay-as-you-go systems. For instance, some countries provide fringe benefits to encourage saving for retirement (e.g., tax deductions, credits, and allowances).[5] The *Savings Credit* of social security in the United Kingdom (UK) was an extra payment for people who had saved up money for retirement. These facts naturally cast some questions on the negative effects of social security on capital accumulation if it is operated as a pay-as-you-go system with additive benefits to induce retirement saving.

Furthermore, two views of public pensions exist in relation to its redistributive effects within pay-as-you-go systems: Beveridgean and Bismarckian schemes.[6] The former has a weak link between individuals' contributions and pension benefits, which exhibits large intragenerational redistribution (e.g., Australia, Ireland, Netherlands, and UK). By contrast, the latter shows less intergenerational redistribution because the individuals' contributions are linked tightly to their retirement benefits (e.g., France, Germany, and Italy). Using an overlapping generations model with three income classes of low, medium, and high income individuals, Conde-Ruiz and Profeta (2007) examined which scheme is chosen under majority voting.[7] Without uncertainty, they showed that low-income individuals prefer a small Beveridgean system, whereas middle-income individuals favor a large Bismarckian system. High-income individuals wish for a fully funded system. Therefore, depending on the density of income classes, either scheme could be chosen politically.

Integrating findings obtained from earlier studies, one finds that the analysis of the effects of social security on economic growth and intragenerational and intergenerational redistribution is fundamentally important to evaluate welfare effects of social security under aggregate and idiosyncratic shocks.[8] To address this, we develop a simple overlapping generations model with social security under the economic-wide productivity and labor supply shocks. We consider social security programs of four types: The fully funded, modified funded, pay-as-you-go (unfunded), and modified unfunded systems, which are related to existing social security programs. For instance, the modified unfunded system in our model is a pay-as-you-go system with Saving Credit in UK, which are fringe benefits that are anticipated as stimulating incentives to save money for old-age consumption. The modified funded system is an intragenerational risk-sharing

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[5] Savings Credit is one type of fringe benefit to attract people to save more. According to the report by European Commission's high-level group of experts on pensions, some countries (e.g., Germany, Croatia, and Italy) apply tax exemption and incentives to encourage personal savings. The tax deduction functions similarly to saving credit here.
[6] Disney (2004) provides details of Beveridgean and Bismarckian schemes.
[7] Numerous studies have addressed this issue (e.g., Casamatta et al., 2000; Cremer and Pestieau, 2000; Cremer et al., 2007; Glasso and Profeta, 2007).
[8] Empirical studies have elucidated a significant relation between economic growth and shocks by postulating risks (e.g., Kormendi and Meguire, 1985; Grier and Tullock, 1989; Ramey and Ramey, 1995; Furceri and Karras, 2007; Imbs, 2007, Alouini and Hubert, 2019). Emphasizing income risks by productivity shocks is important for the examining social security effects.

system: the social security fund manages the pooled revenues of social security taxes and pays pension benefits from the pooled funds. Therefore, it differs from personal savings. It is fully funded based on personal pension accounts.

First, we demonstrate that the equilibrium growth rate under a modified unfunded system might be larger than that under a fully funded system, depending on the relative risk aversion. A savings credit gives people an incentive to save more. With low relative risk aversion, people might prefer saving because it makes their future returns become large. They also benefit from large returns with risks. For that reason, a modified unfunded pension is superior in terms of growth-enhancement. However, with high relative risk aversion, people wish to avoid risky behavior. The growth rate under the modified unfunded system becomes the second-lowest after pay-as-you-go.

Second, we derive the welfare effects of different pension systems. Under the Rawlsian welfare function, the social welfare level under the fully funded pension system is the lowest because it provides no benefit for the poorest people. By contrast, the modified funded system will improve social welfare, although its economic growth rate does not differ from that achieved under the fully funded system. Furthermore, a fully compulsory pension managed by the modified funded system is expected to have the highest welfare level because the poorest people obtain consumption opportunities without risks. The pay-as-you-go system with optimal interior social security tax rate generates a higher welfare level than a fully funded system. Furthermore, pay-as-you-go is superior to modified unfunded with fringe benefits at the welfare level, except for small relative risk aversion.

Third, numerical analyses confirm the theoretical findings and quantitative implications. The results provide illustrative examples of theoretical findings. Particularly if people are highly risk-averse, they will choose pay-as-you-go irrespective of the type of social welfare function and distribution of labor endowments. Under the Benthamite welfare function, the distribution of labor endowments affects the optimal social security tax under pay-as-you-go. The thick population of the low-income class tends to increase the optimal tax rate.

The remainder of this paper is organized as follows. Section 2 presents a description of the basic setup of our model. Section 3 examines equilibrium properties of economies with the funded system and also provides equilibrium analyses of economies with the unfunded system. Section 4 investigates the relation between risk, economic growth, and social welfare under different pension systems. Furthermore, it provides a discussion about policy implications of our study based on the related literature by particularly addressing the viewpoints of the welfare states. Finally, Section 6 concludes the paper.

2. The model

Consider a closed economy with a homogeneous good. The economy is in discrete time, the time is indexed by subscript t. firms exist as a continuum and the production technology is assumed to be the specification presented by Turnovsky (2000) and Kenc (2004), which includes external effects of capital on labor productivity.[9] The production function of each firm is

$$y_t = X_t k_t, \tag{1}$$

where y_t denotes the firm output at period-t, X_t stands for firm productivity at period t, and k_t represents capital input at period t. We assume that the capital is fully depreciated during one period.

Total factor productivity X_t is a probabilistic variable with probabilistic density function $f(x_t)$; it is independent and identically distributed over time. Each firm faces different productivity shocks even though they are generated by identical stochastic processes. Therefore, investors also face idiosyncratic shocks if security is insufficient to cover the risks. That is, we consider the incompleteness of asset markets, as analyzed by Harenberg and Ludwig (2015). On the other hand, the total factor productivity shocks also affect macroeconomic activity.

Based on the production function (1), we do not treat labor input explicitly for analytical simplicity, although labor is also an input in the production process. As shown in Turnovsky (2000) and Kenc (2004), the workers receive rewards for labor depending on the relative contribution to the average worker. Assuming capital and labor shares are constant over time, we have

$$r_t k_t = \alpha y_t,$$

$$w_t = (1-\alpha) y_t,$$

where r_t denotes the interest rate, w_t signifies the wage payment of each firm, and α represents the capital distribution rate ($0 < \alpha < 1$). Inserting Equation (1) into the above equations yields

$$1 + r_t = \alpha X_t, \tag{2}$$

$$w_t = (1-\alpha) X_t k_t. \tag{3}$$

Individuals live two periods: young and old. We consider a stationary population, similarly to Harenberg and Ludwig (2015). Hence, the population of each generation is normalized to unity. During the young period, the generation born at period t supply Z_t unit of labor, where Z_t

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9 They assume a positive externality of aggregate capital developed by Romer (1986). Bruce and Turnovsky (2013) examine the relation between social security, economic growth, and welfare under lifetime uncertainty using a continuous-time overlapping generations model with the production function similar to ours. By assuming that the production externality arises from the interaction between the aggregate capital-labor ratio, $y_t = X_t k_t^{\alpha} (K_t/L_t)^{1-\alpha} l_t^{1-\alpha}$, where K_t is the aggregate capital stock, l_t is the firm's labor input, and L_t is the aggregate labor input. As it is seen later, the predetermined stock level of capital has no uncertainty. In equilibrium, firm's capital input level coincides with mean of the capital-labor latio. This provides the microeconomic foundation, leading to Equations (1), (2), and (3).

represents a probabilistic variable with probabilistic density function $g(z_t)$ that is independently and identically distributed over time. The young generation receives rewards for labor, depending on h_t, which is $h_t \equiv Z_t/\bar{Z}_t$, and $\bar{Z}_t$ is defined as $\bar{Z}_t \equiv \int_0^\infty z_t g(z_t) dz_t$. Therefore, h_t can be interpreted as worker's productivity, which is also a probabilistic variable following $g(z_t)$. Labor income $w_t h_t$ is allocated to purchasing private goods, paying social security tax, and savings. In the old period, individuals retire and live on savings and public pension benefits. Then, the budget equations for period-*t* generation in the two periods are

$$c_t^y + s_t + \tau_t = w_t h_t, \tag{4}$$

$$c_{t+1}^o = (1 + r_{t+1})s_t + b_{t+1}, \tag{5}$$

where c_t^y denotes private consumption in the young period, s_t represents saving, τ_t stands for the social security tax, c_{t+1}^o expresses private consumption in the old period, and b_{t+1} stands for the pension benefit.

The lifetime utility function for the period *t* generation is

$$U_t = \frac{\left(c_t^y\right)^{1-\theta} - 1}{1-\theta} + \frac{1}{1+\rho}\frac{(c_{t+1}^o)^{1-\theta} - 1}{1-\theta} \text{ for } \theta > 0, \theta \neq 1,$$

$$U_t = \log c_t^y + \frac{1}{1+\rho}\log c_{t+1}^o \text{ for } \theta = 1,$$

where U_t represents the utility level, ρ is the discount rate ($\rho > 0$), and θ denotes the relative risk aversion ($\theta > 0$). Each individual chooses the level of saving to maximize the expected lifetime utility subject to Equations (4) and (5). With pension system ($\tau_t \geq 0$ and $b_{t+1} \geq 0$), the first-order condition becomes

$$\frac{dE[U_t]}{ds_t} = -(w_t h_t - s_t - \tau_t)^{-\theta} + \frac{1}{1+\rho}E\left[R_{t+1}(R_{t+1}s_t + b_{t+1})^{-\theta}\right] = 0, \tag{6}$$

where $R_{t+1} \equiv 1 + r_{t+1}$. Depending on the risks, the social security taxes, and the retirement benefits, Equation (6) gives the individual saving function.

The pension system is operated publicly. We consider pension systems of four types: fully funded (FF), modified funded (MF), pay-as-you-go (PG), and modified unfunded (MU) systems that represent the existing pension schemes. Regardless of the operation methods, the aggregate tax revenue and retirement benefits in period *t* are

$$T_t = \iint \tau_t f(x_t) g(z_t) dx_t dz_t, \tag{7}$$

$$B_t = \iint b_t f(x_t) g(z_t) dx_t dz_t. \tag{8}$$

To consider the capital market equilibrium condition, one must set up details of the pension system. Depending on the pension systems, the budget of public pension is varied: For funded pension systems (FF and MF), the social security tax revenue in period t must be equal to the aggregate retirement benefits in the next period. Because we have $T_t = B_{t+1}$ for FF and MF, under the funded pension system, the capital market equilibrium condition becomes

$$K_{t+1} = S_t + T_t = \iint s_t f(x_t) g(z_t) dx_t dz_t + \iint \tau_t f(x_t) g(z_t) dx_t dz_t, \tag{9}$$

where S_t denotes the aggregate saving.

On the other hand, unfunded pension systems based on a pay-as-you-go principle require that the tax revenue in period t is equal to the aggregate retirement benefits in period t. Therefore, $T_t = B_t$ holds. The capital market equilibrium condition under the unfunded pension system is

$$K_{t+1} = S_t = \iint s_t f(x_t) g(z_t) dx_t dz_t. \tag{10}$$

Going forward, we assume that X_t and Z_t respectively follow lognormal distributions, such that

$$f(x_t) = \frac{1}{\sqrt{2\pi}\sigma_x x_t} \exp\left(-\frac{(\log x_t - \mu_x)^2}{2\sigma_x^2}\right),$$

$$g(z_t) = \frac{1}{\sqrt{2\pi}\sigma_z z_t} \exp\left(-\frac{(\log z_t - \mu_z)^2}{2\sigma_z^2}\right).$$

Lognormality of shocks is used widely in the literature on risk.[10]

Finally, the labor market equilibrium condition requires that the demand for labor is equal to the labor supply. Consequently, we have

$$L_t = \int_0^\infty z_t g(z_t)\, dz_t = \exp\left(\mu_z + \frac{\sigma_z^2}{2}\right) = \bar{Z}.$$

In those equations, $\bar{Z}$ is the aggregate labor supply, which is equal to the mean of labor supply by the normalization of population. Capital and labor market equilibrium conditions derive the resource constraint of this economy. With production technology (1) and (9) or (10), the aggregate economic variables of K_t, Y_t, B_t, and S_t grow at the identical (equilibrium) growth rate. Hereinafter, we specifically examine the case in which the equilibrium growth rate is positive.

3. Social security programs

This section explains details of four pension systems and characterizes the equilibrium with social security programs. In particular, Subsections 3.1 and 3.2 specifically examine funded pension systems. Subsections 3.3 and 3.4 present examination of unfunded pension systems.

[10] Harenberg and Ludwig (2015) considered two aggregate shocks such that a productivity shock a shock to the unit user costs of capital and one idiosyncratic shock that is related to old-age wage. They also assume that the shocks follow a stochastic processes with log-normal distribution.

3.1. Fully funded pension

Fully funded pensions can have two patterns of *expected returns* for individuals: *risky* and *risk-free*. Given that private investment (saving) involves a risky return on investment, individuals will choose different options with regard to risks.

Fully funded pension with announcement of the expected returns (as a benchmark: BM). In Singapore, the Central Provident Funds (CPF) provide a publicly operated fully funded pension. The CPF ensures a rate of return of at least 2.5% for insured persons. The point of this type is that the fund announces the expected returns of the pension. The pension is risk free for insured persons, although there is heterogeneity of their earnings.

Formally, if the government releases the expected returns of public pension as $\bar{R}_{t+1}$, then the budget of the public pension must satisfy

$$b_{t+1} = \bar{R}_{t+1}\tau_t.$$

Accounting for the portfolio selection between private saving and public pension, Equations (4)–(6) yield $s_t = 0$ and the following equation (see Appendix A):

$$\tau_t = \frac{1}{1 + \beta A^\eta} w_t h_t \equiv \xi_{BM} w_t h_t, \tag{11}$$

where

$$\beta \equiv \left(\frac{\alpha^{1-\theta}}{1+\rho}\right)^{-\frac{1}{\theta}}, \eta \equiv \frac{\theta - 1}{\theta}, \log A \equiv \mu_x + \frac{\sigma_x^2}{2}.$$

Using Equations (9) and (11) with $s_t = 0$, the equilibrium growth rate under fully funded pension with risk-free return is

$$\gamma_{BM} \equiv \frac{(1-\alpha)A}{1 + \beta A^\eta} - 1. \tag{12}$$

Fully funded pension with announcement of the distribution of the expected returns (as a usual fully funded pension: FF). Superannuation in Australia is one kind of compulsory savings. It is based on defined contribution pension system. The defined contribution pensions such as Superannuation, 401K, and others are equivalent to personal savings if the risks are not covered by the pension programs. One can suppose that the government announces that the return of the public pension depends on the productivity X_t with the probabilistic density $f(x_t)$. Here, the public pension does not differ from private savings because no disparity exists among the returns:

$$b_{t+1} = R_{t+1}\tau_t.$$

If individuals can optimally choose their private and public (pension) savings, then from Equations (4)–(6), the total saving function is obtained as (see Appendix A)

$$s_t + \tau_t = \frac{1}{1 + \beta A^\eta \lambda^{1-\theta}} w_t h_t \equiv \xi_{FF} w_t h_t, \tag{13}$$

where

$$\log\lambda = \frac{\sigma_x^2}{2}.$$

Equations (9) and (13) lead to the equilibrium growth rate under a fully funded pension:

$$\gamma_{FF} \equiv \frac{(1-\alpha)A}{1+\beta A^{\eta}\lambda^{1-\theta}} - 1. \quad (14)$$

Suppose that A is sufficiently large to ensure a positive rate of economic growth.[11]

The relation between growth rates under fully funded pension. Comparison between Equations (12) and (14) leads to

$$\gamma_{FF} \gtreqless \gamma_{BM} \Leftrightarrow \theta \gtreqless 1.$$

When $\theta = 1$, no difference exists between risky and risk-free returns. The saving rates are identical (i.e., $\xi_{BM} = \xi_{FF}$) because the income and substitution effects of the interest rate change on the youth consumption exactly offset. Alternatively, the income (substitution) effect dominates the substitution (income) effect if $\theta > 1$ $(\theta < 1)$. Given that $E\left[R_{t+1}^{1-\theta}\right]$ takes a smaller (larger) value than $\bar{R}_{t+1}^{1-\theta}$ for $\theta > 1$ $(\theta < 1)$, ξ_{FF} is larger/(smaller) than ξ_{BM}: the total effect of the interest rate change on youth consumption under risky returns is smaller (larger) than that under risk-free returns when $\theta > 1$ $(\theta < 1)$. Therefore, the growth rate under the FF-regime is higher (lower) than that under the BM regime if $\theta > 1$ $(\theta < 1)$.

3.2. Modified funded pension

In any case, the fully funded pension systems presented in the preceding subsection ensure actuarial fairness, but the equilibrium growth rates differ depending on the relative risk aversion. However, the government might wish to use a funded pension to share economic risks during the same generations. In other words, one can consider *the modified funded pension (MF)*, by which individuals pay the social security tax amount depending on their income and receive the same benefits from the reserve. Therefore, even when such a pension is fundamentally funded, no direct link exists between individuals' contribution and their retirement benefits. Then, we have

$$b_{t+1} = R_{t+1}\phi\bar{w}_t\bar{h}_t,$$

where ϕ denotes the social security tax rate on income under the MF regime ($0 \le \phi \le 1$),

$$\bar{h}_t = \bar{Z}^{-1}\int_0^{\infty} z_t g(z_t)dz_t = 1,$$

$$\bar{w}_t = (1-\alpha)\bar{k}_t\int_0^{\infty} x_t f(x_t)dx_t.$$

[11] This assumption covers all cases of economic growth rates under the three pension systems considered herein.

The idea of the MF system could be found in the Japanese modified funded pension system. In Japan, the public pension program started as a fully funded system. However, facing inflation after the World War II, it was amended to a system similar to MF in the sense that no direct link between individuals' contribution and retirement benefits has existed since 1948.[12] With partly defined benefits, the pension system had intragenerational redistribution effects. By facing the shortage of a reserve for public pension, in reality, Japanese pension programs operated on a pay-as-you-go principle at present after changing the program repeatedly. However, it is worthwhile to investigate its intergenerational redistribution effects as another benchmark to fully funded systems.

Individuals predict the pension benefit in the next period as $b_{t+1}^e = \alpha\phi X_{t+1}\bar{w}_t$ because the average value of labor income and productivity depends on the stochastic process. Using Equations (4)–(6), b_{t+1}^e, and $\bar{w}_t$, the savings function under the MF pension becomes (Appendix A)

$$s_t = \frac{(1-\phi)w_t h_t - \beta A^{\eta}\lambda^{1-\theta}\phi\bar{w}_t}{1+\beta A^{\eta}\lambda^{1-\theta}}.$$

The individual saving function is decreasing in both the social security tax rate and benefits. This result is based on Equations (4)–(6). The increased tax rate decreases the disposable income and increases the marginal utility of the young period for the given saving. To maintain equality of marginal utilities, the individual reduces saving. For the given saving, the increased pension benefit reduces the marginal utility of the old period consumption. To equalize the marginal utilities of the young and old periods, savings must be decreased with the pension benefit. The saving effect of the increased tax rate is explained as the combined effects of the increased tax rate and benefit.

The average saving function under the MF pension is

$$\bar{s}_t = \left[\frac{1-\left(1+\beta A^{\eta}\lambda^{1-\theta}\right)\phi}{1+\beta A^{\eta}\lambda^{1-\theta}}\right]\bar{w}_t \equiv \xi_{MF}\bar{w}_t. \tag{15}$$

The intuition of Equation (15) is fundamentally identical to that of the individual saving function. Using Equations (9) and (15) with the pension budget, we obtain the economic growth rate under the modified pension system as

$$\gamma_{MF} \equiv \frac{(1-\alpha)A}{1+\beta A^{\eta}\lambda^{1-\theta}} - 1. \tag{16}$$

Equation (16) is identical to Equation (14). Thereby, we obtain the following.

Lemma 1. $\gamma_{FF} = \gamma_{MF}$.

[12] Yoshihara and Hata (2016) provide a historical review of the Japanese public pension system.

At the aggregate level, the MF pension does not differ from fully funded pension because the government appropriates the social security tax revenue for investment in the capital market and the reserve for paying pension benefits. However, the MF pension has redistributive effects within the same generation at an individual level. From the viewpoint of social security types, fully funded systems such as BM and FF correspond to the Bismarckian scheme, whereas the MU system corresponds to a Beveridgean scheme.

3.3. Pay-as-you-go pension

Based on the *pay-as-you-go principle (PG)*, the social security tax and benefits in period *t* must be balanced within the existing generations. Then, we have

$$T_t = B_t.$$

In many countries, a public pension is aimed at ensuring a certain level of pension replacement rate. Then we assume that it is fixed over time. Consequently, the pension benefit is

$$b_{t+1} = \psi \bar{w}_{t+1} \bar{h}_{t+1},$$

where ψ denotes the fixed replacement rate ($0 < \psi < 1$),

$$\bar{h}_{t+1} = \bar{Z}^{-1} \int_0^\infty z_t g(z_t) dz_t = 1,$$

$$\bar{w}_{t+1} = (1-\alpha) \bar{k}_{t+1} \int_0^\infty x_{t+1} f(x_{t+1}) dx_{t+1}.$$

As stated by Disney (2004), public pension programs are classified to Beveridgean and Bismarckian schemes. With the fixed replacement rate, the pay-as-you-go pension represents a significant departure from actuarial fairness. The pay-as-you-go pension system described in this section corresponds to the Beveridgean scheme, similar to those found in Australia, Ireland, and the UK. Alternatively, we can consider the Bismarckian scheme by which the individuals' contributions link to their retirement benefits. To examine the intragenerational redistribution effects of public pension, we specifically examine Beveridgean schemes such as the UK pension system.

With PG, the capital market equilibrium condition is (10). In the average term, we have $\bar{k}_{t+1} = \bar{s}_t$. Individuals anticipate the future pension benefit as $b^e_{t+1} = (1-\alpha)\psi X_{t+1} \bar{k}_{t+1}$ because they regard the average capital stock as given and the productivity generated from the stochastic process. Then, the saving function under PG is derived from Equations (4)–(6), b^e_{t+1}, and $\bar{k}_{t+1} = \bar{s}_t$ (Appendix B):

$$s_t = \frac{(1-\psi) w_t h_t - \beta \chi \psi A^\eta \lambda^{1-\theta} \bar{s}_t}{1 + \beta A^\eta \lambda^{1-\theta}},$$

where

$$\chi \equiv \left(\frac{1-\alpha}{\alpha}\right).$$

The economic meaning of the saving function under PG is identical to the saving function under a MF pension. Given that the replacement rate is fixed over time, a rise in the replacement rate has the combined effect of the increased tax rate and benefit. This result is parallel to the negative effect of public pension on capital accumulation in the deterministic and stochastic models (e.g., Feldstein, 1974; Hauenschild, 2002; Hillebrand, 2012). In our model, sustainable growth is generated by capital accumulation. Decreasing capital accumulation will negatively affect economic growth (e.g., Yakita, 2001). To see this, we consider aggregate capital accumulation through aggregate saving.

Using the individual saving function, the average saving function under unfunded pension is (Appendix B)

$$\bar{s}_t = \left[\frac{1-\psi}{1+(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}}\right]\overline{w}_t \equiv \xi_{PG}\overline{w}_t. \tag{17}$$

The PG social security program has two negative effects on saving through the distortionary effect of income tax and saving adverse effect of retirement benefits. Particularly, the latter effect is affected by the risks and relative risk aversion. The effect of the interest factor under risks appears in the deflator of Equation (17) as well as Equation (14), although it is strengthen by the presence of the saving adverse effect in case of PG. Furthermore, larger θ engenders the strong saving adverse effect. The relative risk aversion determines the sensitivity of economic growth change in response to risks. The average saving rate decreases with the fixed replacement rate:

$$\frac{\partial \xi_{PG}}{\partial \psi} = -\frac{1+(1+\chi)\beta A^{\eta}\lambda^{1-\theta}}{[1+(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}]^2} < 0.$$

The economic intuition follows the result obtained for individual saving.

Using Equations (10) and (17), we obtain the economic growth rate under PG, such that

$$\gamma_{PG} \equiv \frac{(1-\psi)(1-\alpha)A}{1+(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}} - 1. \tag{18}$$

Partial derivation of Equation (18) with respect to ϕ yields

$$\frac{\partial \gamma_{PG}}{\partial \psi} = -\frac{(1-\alpha)\left[A+(1+\chi)\beta A^{1+\eta}\lambda^{1-\theta}\right]}{[1+(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}]^2} < 0.$$

Because the capital accumulation depends on saving, the replacement rate influences the economic growth rate through saving. As demonstrated, an increase in the replacement rate has a negative effect on saving because of the distortionary effect of income tax and saving adverse effect of retirement benefits. Therefore, the economic growth rate is negatively associated with the replacement rate. Alongside this result and Lemma 1, comparing Equations (13) and (18), one obtains

Lemma 2. $\gamma_{PG} \le \gamma_{FF}$ *for* $\psi \ge 0$.

Irrespective of the value of θ, the economic growth rate under PG pension is dominated by that under the fully funded or MF pensions. When $\psi = 0$, $\gamma_{PG} = \gamma_{FF} = \gamma_{MF}$ holds. In principle, the pension benefit decreases private saving. Furthermore, the pension benefits of PG derive from the social security tax of the next generation, which is not used in the capital market. The growth rate under PG is decreasing in the replacement rate. This result is the same as that of the model without risk. However, the risks quantitatively affect savings and economic growth.

3.4. Pay-as-you-go pension with additive benefits to stimulate saving incentives

Earlier, we considered the PG pension designed in a conventional manner. As demonstrated in earlier studies and ours, PG discourages saving for retirement. Some countries have adopted fringe benefits to encourage saving for retirement (e.g., tax deduction, credit, and allowance). For instance, *Savings Credit* of the public pension in the United Kingdom was an extra payment for people who had saved up money for retirement. To address such a pension system in reality, we consider *the modified unfunded pension (MU)*, which gives individuals incentive for saving.

Given that savings credit is positively associated with private saving relative to average saving, the pension benefit is formalized as

$$b_{t+1} = \left[\pi + (1-\pi)\frac{s_t}{\bar{s}_t}\right]\psi \bar{w}_{t+1}\bar{h}_{t+1},$$

where $(1-\pi)$ denotes the parameter related to savings credit ($0 < \pi < 1$). This can be interpreted as one way to introduce actuarial fairness. When $\pi = 1$, the modified pay-as-you-go pension coincides with the standard pay-as-you-go system follows the Beveridgean manner. In contrast, the modified pay-as-you-go system corresponds to unfunded but fund-like pension with the returns as the economic growth rate if $\pi = 0$. Therefore, MU such as $\pi = 0$ is one representative of the Bismarckian scheme.[13] The pension budget is the same as $T_t = B_t$ at the aggregate level.

Equations (4)–(6) engender the following (Appendix B):

$$s_t = \frac{(1-\psi)w_t h_t - \omega\beta\pi\chi\psi A^{\eta}\lambda^{1-\theta}\bar{s}_t}{1 + (1 + (1-\pi)\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}},$$

[13] From this viewpoint, π can be interpreted as the Bismarckian factor, which is formulated by Conde-Ruiz and Profeta (2007). However, our specification is more specified to the existing pension program.

where $\omega \equiv (1+(1-\pi)\chi\psi)^{-\frac{1}{\theta}}$. $0<\omega<1$ holds. The average saving function becomes (Appendix B)

$$\bar{s}_t = \left[\frac{1-\psi}{1+(1+\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}}\right]\bar{w}_t \equiv \xi_{MU}\bar{w}_t. \tag{19}$$

By the same manner of interpreting Equation (19), the MU system has two negative effects on saving: the distortionary effect of income tax and saving adverse effect of retirement benefits. The most attractive point is that the saving credit affects the saving adverse effect of retirement benefits; it will weaken the saving adverse effect. Using Equations (10) and (19), the economic growth rate under MU pension system is

$$\gamma_{MU} \equiv \frac{(1-\psi)(1-\alpha)A}{1+(1+\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}} - 1. \tag{20}$$

We now characterize the effects of the unfunded pension with saving-induced fringe benefits on economic growth rate. Partial differentiation of Equation (20) regarding ψ provides

$$\frac{\partial\gamma_{MU}}{\partial\psi} = -\frac{\left[1+(1+\chi)\omega\beta A^{\eta}\lambda^{1-\theta}\right]+(1-\psi)(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}\frac{\partial\omega}{\partial\psi}}{\left[1+(1+\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}\right]^2}(1-\alpha)A,$$

where

$$\frac{\partial\omega}{\partial\psi} = -\frac{(1+(1-\pi)\chi\psi)^{-\frac{1}{\theta}-1}(1-\pi)\chi}{\theta} < 0.$$

Unfunded pension with saving-induced fringe benefits might enhance economic growth because the growth effect of a rise in ψ through a change in ω is positive.

When $\psi = 0$, the marginal growth effect is

$$\left.\frac{\partial\gamma_{MU}}{\partial\psi}\right|_{\psi=0} = -\frac{\left[1+(1+\chi)\beta A^{\eta}\lambda^{1-\theta}\right]-\frac{(1-\pi)\chi\beta A^{\eta}\lambda^{1-\theta}}{\theta}}{(1+\beta A^{\eta}\lambda^{1-\theta})^2}(1-\alpha)A.$$

For a small θ, a rise in ψ increases the growth rate at $\psi = 0$. Contrarily, the effect of ψ on growth rate at $\psi = 1$ is negative, as

$$\left.\frac{\partial\gamma_{MU}}{\partial\psi}\right|_{\psi=1} = -\frac{1}{1+(1+\chi)\omega\beta A^{\eta}\lambda^{1-\theta}} < 0.$$

Therefore, the social security tax and economic growth rate have a hump-shaped (monotonically decreasing) relation if θ is sufficiently small (large). The reported values of θ by empirical studies are varied over the range of 0.2–10. The most widely accepted value of θ would be between 1 and 3 (Gandelman and Hernandez-Murillo, 2015). Gandelman and Hernandez-Murillo (2015) found that some economically developed countries have values smaller than 0.5 (e.g., Ireland, Japan, Korea, and the Netherlands).[14] Sufficiently small values of θ would be plausible.

We next consider the effects of a change in π on economic growth. The partial derivative of Equation (20) regarding π yields

$$\frac{\partial \gamma_{MU}}{\partial \pi} = -\frac{(1-\psi)(1-\alpha)\chi\psi\beta A^{1+\eta}\lambda^{1-\theta}}{[1+(1+\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}]^2}\frac{\partial \omega}{\partial \pi} < 0.$$

The equation above demonstrates that small (large) saving credits engender a low (high) economic growth rate.

Equations (13) and (20) show that the properties of equilibrium growth rate, and Lemmas 1 and 2 provide the following (Appendix C provides the proof):

Proposition 1. *(i) When* θ *is sufficiently small, there exists* $\hat{\psi}$ *such that* $0 < \hat{\psi} < 1$ *and* $\gamma_{MU} = \gamma_{FF}$. *Then,* $\gamma_{PG} < \gamma_{FF} < \gamma_{MU}$ *if* $\psi < \hat{\psi}$ *while* $\gamma_{PG} < \gamma_{MU} \leq \gamma_{FF}$ *if* $\psi \geq \hat{\psi}$. *(ii) When* θ *is sufficiently large,* $\gamma_{PG} < \gamma_{MU} < \gamma_{FF}$ *holds for* $\psi > 0$.

Small θ denotes that the consumptions between youth and old age are more substitutable. The income shocks have less effect on consumption-saving choice than large θ. Given that other economic conditions are unchanged, saving under small θ is less than under large θ. Therefore, the growth rate under fully funded pension will be larger than under PG because the latter induces people to consume more. For small θ, unfunded pension with saving-induced fringe benefits might accelerate economic growth by stimulating saving compared to a fully funded pension. For small saving under small θ, strengthening saving incentive will positively affect economic growth. The MU pension system increases the economic growth rate over the level under full funding within the appropriate values of ψ.

4. Growth and welfare effects of risks through public pension

A tradeoff between economic growth and social justice is a central issue of public economics. If no transitional dynamics or no heterogeneity of economic agents with the production function (1) exists, then the maximizing growth rate can be consistent with attaining social justice. However, this will not be true with income shocks. To consider the tradeoff between economic growth and social justice, this section presents examination of the growth and welfare effects of income risks through different public pension systems.

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[14] The values of θ are 0.35 in Ireland, 0.44 in Japan, 0.27 in Korea, and 0.10 in the Netherlands. θ equals 2 is rejected at the 10 percent level for all countries and $\theta = 1$ is also rejected at the 10 percent level for Korea.

4.1. Risks and economic growth

Following conventional methods, risks should be measured as variances. Parameters related to risks are μ_i and σ_i $(i = x, z)$. Both μ_i and σ_i affect the mean and variance of X_t or Z_t. The distribution of labor ability only influences individual saving and does not aggregate saving. The variance $V[X_t]$ is zero if $\sigma_x = 0$. Herein, we specifically examine the comparative statics of σ_x.

Taking a logarithmic differentiation of Equations (13) and (15) for σ_x, the elasticities of the growth factors in the funded pension systems for σ_x become

$$\epsilon_{FF} \equiv \frac{\sigma_x \partial \log(1+\gamma_{FF})}{\partial \sigma_x} = \left[1 - \frac{(\theta-1)^2}{\theta}\left(\frac{\beta A^\eta \lambda^{1-\theta}}{1+\beta A^\eta \lambda^{1-\theta}}\right)\right]\sigma_x^2,$$

$$\epsilon_{MF} \equiv \frac{\sigma_x \partial \log(1+\gamma_{MF})}{\partial \sigma_x} = \epsilon_{FF}.$$

Similarly, the logarithmic differentiations of Equations (18) and (20) for σ_x lead to

$$\epsilon_{PG} \equiv \frac{\sigma_x \partial \log(1+\gamma_{PG})}{\partial \sigma_x} = \left\{1 - \frac{(\theta-1)^2}{\theta}\left[\frac{(1+\chi\psi)\beta A^\eta \lambda^{1-\theta}}{1+(1+\chi\psi)\beta A^\eta \lambda^{1-\theta}}\right]\right\}\sigma_x^2,$$

$$\epsilon_{MU} \equiv \frac{\sigma_x \partial \log(1+\gamma_{MU})}{\partial \sigma_x} = \left\{1 - \frac{(\theta-1)^2}{\theta}\left[\frac{(1+\chi\psi)\omega\beta A^\eta \lambda^{1-\theta}}{1+(1+\chi\psi)\omega\beta A^\eta \lambda^{1-\theta}}\right]\right\}\sigma_x^2.$$

We now consider the marginal effects of increased risk on the growth effects from σ_x. Comparison between the elasticities results in the following.

Proposition 2. *(i)* $(1+\chi\psi)\omega > 1 \Leftrightarrow \epsilon_{MU} < \epsilon_{PG} < \epsilon_{FF}$, *while (ii)* $(1+\chi\psi)\omega < 1 \Leftrightarrow \epsilon_{PG} < \epsilon_{FF} < \epsilon_{MU}$.

A rise in σ_x increases not only the average productivity but also the income risks. The overall effect on economic growth is varied, depending on the schemes and preference parameters. Specifically examining $\sigma_x = 0$, we have

$$(1+\chi\psi)\omega \gtreqless 1 \Leftrightarrow \theta \gtreqless \frac{\log(1+(1-\pi)\chi\psi)}{\log(1+\chi\psi)} \equiv \hat{\theta},$$

where $\hat{\theta} < 1$. For example, we consider $\theta > \hat{\theta}$. We obtain $\epsilon_{MU} < \epsilon_{PG} < \epsilon_{FF}$ from Proposition 2. Presuming that $\epsilon_{FF} < 0$ holds, then $|\epsilon_{MU}| > |\epsilon_{PG}| > |\epsilon_{FF}| > 0$ because of $\epsilon_{MU} < \epsilon_{PG} < \epsilon_{FF} < 0$. The unfunded pension with saving credits exhibits the highest sensitivity of economic growth to increased risk. However, if we assume $\epsilon_{MU} > 0$ and $|\epsilon_{FF}| > |\epsilon_{PG}| > |\epsilon_{MU}| > 0$ holds, then the fully funded pension shows the highest sensitivity of economic growth to increased risk. The relation between risk and economic growth can be elucidated using numerical simulations.

4.2. Optimal pension system

Using Equations (4), (5), and (14), the indirect utility function under a fully funded pension with risky returns is

$$U_t^{FF}=\frac{\left((1-\psi_{FF})w_th_t\right)^{1-\theta}-1}{1-\theta}+\frac{1}{1+\rho}\frac{E\left[(R_{t+1}\psi_{FF}w_th_t)^{1-\theta}\right]-1}{1-\theta}.$$

Similarly to deriving W_t^{FF}, we obtain the indirect utility functions under MF pension such that

$$U_t^{MF}=\frac{\left\{\frac{[(1-\phi)w_th_t+\phi\bar{w}_t]\beta A^\eta\lambda^{1-\theta}}{1+\beta A^\eta\lambda^{1-\theta}}\right\}^{1-\theta}-1}{1-\theta}$$
$$+\frac{1}{1+\rho}\frac{E\left[\left(\frac{(1-\phi)w_th_t+\phi\bar{w}_t}{1+\beta A^\eta\lambda^{1-\theta}}R_{t+1}\right)^{1-\theta}\right]-1}{1-\theta},$$

$$U_t^{PG}=\frac{\left[\frac{(1-\psi)\beta A^\eta\lambda^{1-\theta}w_th_t+\beta\chi\psi A^\eta\lambda^{1-\theta}\bar{s}_t}{1+\beta\lambda^{1-\theta}A^\eta}\right]^{1-\theta}-1}{1-\theta}$$
$$+\frac{1}{1+\rho}\frac{E\left[\left(\frac{(1-\psi)w_th_t-\beta\chi\psi A^\eta\lambda^{1-\theta}\bar{s}_t}{1+\beta A^\eta\lambda^{1-\theta}}R_{t+1}+\psi\bar{w}_{t+1}\bar{h}_{t+1}\right)^{1-\theta}\right]-1}{1-\theta},$$

$$U_t^{MU}=\frac{\left\{\frac{[(1-\psi)w_th_t](1+(1-\pi)\chi\psi)\omega\beta A^\eta\lambda^{1-\theta}+\omega\beta\pi\chi\psi A^\eta\lambda^{1-\theta}\bar{s}_t}{1+(1+(1-\pi)\chi\psi)\omega\beta A^\eta\lambda^{1-\theta}}\right\}^{1-\theta}-1}{1-\theta}$$
$$+\frac{1}{1+\rho}\frac{E\left[\left(R_{t+1}s_t+\left[\pi+(1-\pi)\frac{s_t}{\bar{s}_t}\right]\psi\bar{w}_{t+1}\bar{h}_{t+1}\right)^{1-\theta}\right]-1}{1-\theta}.$$

We assume that the social welfare function takes the form of the Rawlsian welfare function. Formally, the social welfare function under pension scheme *i* ($i=FF,MF,PG,MU$) is defined as

$$W_t^i\equiv\sum_{T=t}^{\infty}\left(\frac{1}{1+\delta}\right)^T\min U_T^i,$$

where δ denotes the social discount rate. $\gamma_i<\delta$ ($i=FF,MF,PG,MU$) is necessary to ensure that the social welfare is bounded. Other welfare criteria can be presumed, such as the Benthamite welfare function:

$$V_t^i\equiv\sum_{T=t}^{\infty}\left(\frac{1}{1+\delta}\right)^T\iint U_T^i\ f(x_T)g(z_T)dx_Tdz_T.$$

However, deriving analytical results other than the Rawlsian welfare function is complicated. Therefore, we specifically examine the Rawlsian welfare function, although the Benthamite social welfare function will be analyzed numerically.

The welfare effect on W_t^{MF} consists of effects on indirect utility and economic growth. As shown earlier, a rise in ϕ has no effect on economic growth. Hence, the welfare effect of a change in ϕ is

$$\operatorname{sgn}\left(\frac{\partial W_t^{MF}}{\partial \phi}\right) = \operatorname{sgn}\left(\frac{\partial \min U_t^{MF}}{\partial \phi}\right)$$

$$= \phi^{-\theta}\left\{\left(\frac{\beta A^{\eta}\lambda^{1-\theta}\overline{w}_t}{1+\beta A^{\eta}\lambda^{1-\theta}}\right)^{1-\theta} + \frac{1}{1+\rho}E\left[\left(\frac{R_{1+t}^{1-\theta}\overline{w}_t}{1+\beta A^{\eta}\lambda^{1-\theta}}\right)^{1-\theta}\right]\right\} > 0.$$

The optimal social security tax rate on income under the MF regime is equal to its upper limit. When $\phi = 0$, $W_t^{MF} = W_t^{FF}$ holds. We obtain $W_t^{MF} \geq W_t^{FF}$ for $\phi \geq 0$. Within the funded pension systems, these results lead to the following.

Proposition 3. *For $\phi > 0$, W_0^{MF} is larger than W_0^{FF}. The optimal social security tax rate of MF pension is $\phi^* = 1$.*

We turn to the analysis of the welfare effects of the unfunded pension systems. Within the unfunded pension systems, the social security tax rates affect the economic growth rates. Therefore, the increased pension replacement rate influences social welfare through short-term and long-term effects: the former involves intergenerational redistribution and tax burden effects; the latter is based on the negative growth effects of public pension. As mentioned in the previous sections, the relative risk aversion affects the negative growth effect because saving adverse effect of retirement benefits are weakened or strengthened depending on θ. Therefore, the welfare effects of the unfunded social security programs and the optimal system are varies on θ. Considering these effects, we obtain the following proposition (Appendix D provides the proof):

Proposition 4. *There exists an interior optimal social security tax rate of PG pension, ψ^*. If θ is sufficiently small, there might exist an interior optimal saving credit rate π^*. Contrarily, PG is optimal if θ is sufficiently large. In either case, W_0^{MU} is equal to or greater than W_0^{PG} for the optimal saving credit rate.*

People with no income do not benefit from economic growth at present because they cannot consume and save. They are willing to receive pension benefits. Then, governments must care about the negative growth effects of PG because future pension benefits depend on the next young generations' income levels (ultimately economic growth rate). Therefore, the social security tax rate is optimally selected as an interior solution irrespective of the parameter values. However, the optimal tax rate of social security depends on θ, σ_x, and so on.

With regard to the optimal saving credit rate, a government might choose either an optimal interior credit rate or no saving credit (i.e., PG pension system). If $\pi = 0$, then it means no pension for the poorest people. Therefore, the government has an incentive to increase π and decrease saving credit because the current effect on the poorest people includes the redistributive effect of PG if $\pi > 0$. For $\pi > 0$, the poorest person's utility depends on the economic growth rate. The governments aim not to decrease the economic growth rate too much and might keep it high to set $\pi < 1$. This is true if θ is sufficiently low. However, if θ is sufficiently large, then no saving credit (i.e., $\pi = 1$) is preferable because high growth and risks with large saving credit negatively affect utility level.

4.3. Numerical analysis

We set the parameters as $\alpha = 0.3$ and $\rho = 0.5$.[15] Following the estimated result presented by Gandelman and Hernandez-Murillo (2015), the key parameter θ is considered at three different values: 0.5, 1, and 1.5.[16] Furthermore, $\mu_z = 0$, and $\sigma_z = (0.5, 1, 3, 5, 10)$. Microeconomic data for firms and households are necessary to identify parameters related to idiosyncratic productivity shocks. However, our specified model requires specified data and must overcome a lack of the data. This numerical analysis aims at providing certain examples to show theoretical predictions in the preceding sections. To avoid difficulties of data treatment, we specifically examine the case in which idiosyncratic shock parameters correspond to macroeconomic shocks parameters one to one. Therefore, the economy consists of representative economic agents with identical probability density functions. Although incompleteness of our estimation exists, it is apparently sufficient for giving the parameters which generate realistic equilibrium values.

The parameters related to productivity shocks should be calculated using observed economic variables. Using Equations (1) and (16), the relation between productivity and observable economic variables satisfies

$$X_{t+1} = \frac{Y_t}{S_t} \times \frac{Y_{t+1}}{Y_t} = \frac{\text{Growth factor}}{\text{Investment-to-output ratio}}.$$

Using the US data of GDP and investment from 1947Q1 to 2020Q4, we calculated the value of

[15] The capital share lies between 0.3 and 0.5. Therefore, the value around $\alpha = 0.3$ is used frequently for numerical analysis. For instance, Imrohoroglu et al. (1998) set $\alpha = 0.36$ to calibrate the effects of individual retirement accounts on capital accumulation.

[16] Based on estimation of Gandelman and Hernandez-Murillo (2015), the density is quite low, out of the range of 0.5–1.5. Therefore, we specifically examine these three values as representatives.

X from 1947Q1 through 2020Q4.[17]

The distribution parameters during all data periods are estimated as $(\mu_x, \sigma_x) = (2.025, 0.187)$. The Anderson–Darling statistic is 3.199 (p-value: 0.022). Consequently, the estimated parameter is statistically not significant. This non-significance might derive from large shocks and structural changes in the distribution parameters. We divide the time periods into four: first Period (1950Q1–1975Q4), Second Period (1976Q1–1997Q4), Third Period (1998Q1–2008Q4), and Fourth Period (2009Q1–2020Q4). Estimation of the First Period yields $(\mu_x, \sigma_x) = (2.200, 0.092)$, where the Anderson–Darling statistic is 0.282 (p-value: 0.951). These parameter values are significant at the 5% level, whereas no other period was found to be significant at the 5% level.[18] Based on our estimated results, we set the productivity shock parameters as $\mu_x = 2$ and $\sigma_x = (0, 0.25, 0.5, 0.75, 1)$.

Economic growth, public pension, and risks. Actually, $\psi = 0.2$ and $\pi = 0.9$ are assumed for calculating the equilibrium growth rate under the PG and MU systems. Tables 1 and 2 present the equilibrium growth rates. In Table 1, a comparison between benchmark and fully funded/MF gives the numerical example of the results from Section 3. Table 2 demonstrates that the growth rate under MU is larger than that under PG for each case (Proposition 1). Comparison between the results Tables 1 and 2 indicates that a fully funded/MF pension generates the highest growth rate. Therefore, within plausible values of parameters, fully funded/MF is the best way to boost economic growth.

The MU system stimulates economic growth through saving credit, but its effect is too weak to overweigh that of the funded system within the value of $0.5 \leq \theta \leq 1.5$ for $\alpha = 0.3$. However, when $\alpha = 0.2$ and $\theta = 0.5$, the growth rate under MU is larger than that under FF/MF for $0 < \psi < 0.292$. For small value of α, the MU system generates economic growth rate higher than the FF/MF system within the plausible rage of the relative risk aversion. The recent trend for declining labor share may weaken the advantage of MU to FF/MF regarding economic growth.

Social welfare, public pension, and risks. With regard to economic growth, funded pension systems are superior to unfunded pension systems. However, under the Rawlsian welfare function, the fully funded pension system brings about the lowest welfare level for the poorest workers because no income derives no saving for consumption in retirement. Therefore, we should quantitatively verify the order of welfare levels and the optimal social security. The labor supply shocks do not affect the optimal tax rates under the Rawlsian welfare function.

We set $\delta = 1$ as the social discount rate. Table 3 demonstrates that the optimal social security tax rate is equal to unity under the MF system because the poorest workers benefit from a public

[17] U.S. Bureau of Economic Analysis, Real Gross Domestic Product [GDPC1] and Real Gross Private Domestic Investment [GPDIC1], retrieved from FRED, Federal Reserve Bank of St. Louis.

[18] The estimated parameter values are (2.029, 0.079) for the 2nd Period, (1.793, 0.039) for the 3rd, and (1.819, 0.123) for the 4th. The Anderson–Darling statistics are 0.589 (p = 0.658) for the 2nd Period, 0.520 (p = 0.726) for the 3rd, and 2.289 (p = 0.064) for the 4th.

pension with no tax burden (Proposition 3). By contrast, PG outcomes indicate that an optimal interior social security tax rate exists. For $\theta = 0.5$, the tax rate range of 4% and 15%. For $\theta = 1$ and $\theta = 1.5$, the tax rates are around 20%. Observation of PG outcomes indicates that larger σ_x engenders a smaller tax rate. An increase in σ_x raises the expected productivity and its volatility. It increases the interest and economic growth rates. These effects diminish the need for raising social security tax rate by increasing pension benefits.

In Table 3, MU can be superior to PG for $\theta = 0.5$, although PG is preferable (i.e., $\pi^* = 1$) for $\theta = 1$ and $\theta = 1.5$. When $\theta = 0.5$, the optimal percentage of saving credit is 78.9% – 86.3% for the domain of σ_x (Proposition 4). Therefore, the unfunded system with saving credit is theoretically justified in certain cases. For small θ, high growth and interest rates by increased risks improve welfare. The poorest households seek to benefit from increasing interest and economic growth rates. Therefore, the saving credit is optimally selected.

Regarding the welfare level, Table 3 demonstrates that MF generates the highest welfare level. When $\theta = 1$ and $\theta = 1.5$, no saving credit ($\pi = 1$) is selected; PG is in second place. However, if $\theta = 0.5$, then the order is reversely changed. The MU system is a better choice by the intuition of Proposition 4.

Social welfare, public pension, and population density of income class. Based on computations, any class of specified welfare function gives optimal social security tax rate and its welfare level. To verify the robustness of our guess related to the optimal social security tax rate, we consider the Benthamite welfare function V_t^i. Parameter σ_x is fixed at $\sigma_x = 0.5$, whereas σ_z is varied in $\sigma_z = (0.5, 1, 3, 5, 10)$. The density of each income class determines the optimal social security tax rate. Small σ_z indicates that the income distribution is not widely spread, whereas large σ_z indicates that the low-income people have a thick population.

Table 4 presents the calibrated results. The outcomes under the Benthamite welfare function exhibit similarities to those obtained under the Rawlsian welfare function: social security tax rates and the orders of welfare levels. However, when $\theta = 0.5$ and $\sigma_z = 0.5$, welfare under the MU system is greater than that under MF. The table demonstrates that the optimal social security tax rate is increasing in σ_z. High density of low-income class engenders high tax rate. Furthermore, high relative risk aversion tends to increase social security tax rate similarly to that in Table 3.

4.4. Discussion

This subsection provides discussion of the policy implications of our study for existing *Welfare States* and further analyses for endogenous fertility.

Policy implication for Existing Welfare States. Our theoretical and numerical analyses reveal that an unfunded pension is preferred to a fully funded pension if the government cares about poor people. Rich people naturally prefer to buy private annuities, which are identical to a fully funded pension. However, for various political reasons, the governments in the real world must devote attention to low-income people. Therefore, the government operates the public pension on a pay-as-you-go principle.

Our results also indicate that the modified funded pension system has some advantages over a pay-as-you-go system and fully funded pension systems in response to risks. Although providing a higher welfare level than others, this system loses the link between contributions and retirement benefits. In reality, it might be difficult to obtain public agreement for the modified funded system. Consequently, we conclude that the unfunded pension system might be a better choice in an economy with risks, which explains why unfunded pension systems are mainstream to provide social security programs.

Within unfunded pension systems lies the issue of intragenerational redistribution. As described earlier, real social security programs are classified as Beveridgean and Bismarckian pension systems. Using the political economic approach, some studies have tackled issues of which of them is politically chosen (e.g., Casamatta et al., 2000; Cremer and Pestieau, 2000; Conde-Ruiz and Profeta, 2007; Cremer et al., 2007; Glasso and Profeta, 2007). They demonstrated that the Beveridgean system is politically supported by low-income people, although Bismarckian systems are favored by high-income people, and demonstrated that which system arises depending on the retirement timing, demographics, and so on.[19]

In our study, qualitative and quantitative analyses demonstrate that the Beveridgean system such as PG is preferred if people tend to avoid risks strongly, whereas the Bismarckian system such as MU is favored when people tend to weakly avoid risks. In fact, this result is similar to that described earlier in the literature on Beveridgean and Bismarckian pension systems because the relative risk aversion can be interpreted as the relative inequality aversion. However, our numerical results imply a difference of welfare implications in response to risks of different types.

The degree of saving credit, π, decreases with the variance of labor productivity directly related to earning, σ_z; a large variance of the worker's productivity engenders the Beveridgean. Furthermore, whether a small or large pension system is better depends on the labor-related risk and social welfare function. For given the firm's productivity, large inequality (i.e., large value of σ_z) meets a large pay-as-you-go pension. However, different patterns are visible for the firms' productivity shocks. A large variance of the firm's productivity, σ_x, engenders small pay-as-you-go pensions. Because the firm's productivity is linked to aggregate productivity, which determines the equilibrium economic growth rate, a large value of σ_x generates a high growth rate with high

[19] Kaganovich and Zilcha (2012) examined the fiscal sustainability of social security including the public education funding. They demonstrate that that the fully funded social security system generates political support for a higher education funding and therefore a higher economic growth rate than the pay-as-you-go system.

risks. It requires reduction of a distortionary effect of a tax. The optimal social security tax rate decreases with σ_x.

These results imply that the desirable pension systems are faced with different levels of risk and relative risk aversion. In reality, almost all economically developed countries have adopted pay-as-you-go pension systems between Beveridgean and Bismarckian schemes. Based on our analyses, if people are strongly risk or inequality averse, the society tends to prefer large intragenerational redistribution within pay-as-you-go pension systems. The Bismarckian factor averages in 1988–2008 are 0.05 for Australia, 0563 for Germany, 0.341 for Ireland, 0.307 for the Netherlands, and 0.127 for UK (Rivera-Rozo et al., 2018), whereas the corresponding values of the relative risk aversion are 1.17 for Australia, 0.77 for Germany, 0.35 for Ireland, 0.1 for Netherland, and 1.03 for United Kingdom (Gandelman and Hernandez-Murillo, 2015). These tendencies support our result that large θ engenders large π. Therefore, our model gives a plausible explanation about the existing Welfare States.[20]

Further analyses. As described in this paper, we treat labor supply shocks to consider heterogeneity of workers' earnings. Because the population of the economy is stationary, there is no demographic change. However, if the variance of exogenous labor supply is changed, then the aggregate labor supply is also changed. Such a shock can be interpreted as a shock for the working population. Therefore, our results are applicable to explain the effects of exogenous change in working populations on equilibrium outcomes through social security programs.

Demographic change can be generated by changes in economic circumstances such as income. Therefore, the demographic shocks are not independent of labor supply and are one determinant of fertility. Considering endogenous determination of the numbers of children, certain studies show that social security systems affect fertility rates (e.g., Cigno, 1993; Zhang, 1995; van Grozen et al., 2003; Sinn, 2004). Although this issue is fundamentally important to analyze the demographic shocks in relation to fertility and mortality rates, we can guess the feedback effects of social security on fertility using the theoretical findings presented in earlier studies of the literature.

Numerous researchers have examined the relation between population aging, social security, and economic growth using overlapping generations model with endogenous fertility (e.g., Pecchenino and Pollard, 1997; Yakita, 2001; Hirazawa et al., 2010; Ono, 2017), which enables us to analyze population aging and its economic effect on (or through) social security. However, separate from fertility choice, demographic shocks have not been specifically examined in endogenous growth models. The different demography in each generation causes asymmetric labor supply and intragenerational and intergenerational income inequality. Furthermore,

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[20] Rivera-Rozo et al. (2018) found that the Bismarckian factor is affected by cultural factors (e.g., individualism) as same as economic factors. Therefore, our model cannot all the cases of the existing Welfare States. For example, the Bismarckian factor and relative risk aversion in United States are 0.489 and 1.39 respectively. To explain this, we must incorporate additional factors into our basic model.

productivity shocks are significant with regard to economic growth. Our theoretical findings fill the gap in the research.

When the number of children is formulated as one of consumption goods and normal goods, a decrease in income decreases the number of children. With large relative risk aversion, an increase in the variance of labor productivity will decrease the number of children. It negatively affects the revenue of social security taxes under pay-as-you-go pension systems and therefore might decrease social welfare. In contrast, the firms' productivity risks will have different effects on fertility. An increase in variance of the firms' productivity increases the economy-wide productivity and therefore raises average wages and interest rates with large volatilities of them. Depending on the relative risk aversion, these increases in wages, interest rates, and their volatilities affect the expected utility in the second period. However, the basic mechanism revealed by our study will provide an analytical basis for the extensions of endogenous fertility.

5. Conclusion

This paper presented an examination of the relation between public pensions, economic growth, and social welfare in an overlapping generations model with income shocks. We considered four pension systems for comparison: FF, MF, PG, and MU. Particularly, MF and MU are characteristics of this study. The former provides a risk-pooling function, which involves income redistribution effects within the same generations, whereas the latter gives people incentives for saving by fringe benefit.

First, we analyzed the relation between economic growth and public pension. The equilibrium growth rate under the fully funded system coincides with that under modified pensions. Actually, PG generates the lowest growth. The results obtained for the fully funded and PG regime complement those of earlier studies. The main contribution of this study is the role of fringe benefits in providing saving incentives, indicating that the equilibrium growth rate under MU might be higher than that under full funding, depending on the relative risk aversion. Because private saving behavior is affected by risks and fringe benefit, they can stimulate saving and economic growth.

Next, we investigated the relation between social welfare and public pension. Specifically examining the Rawlsian welfare function, the social welfare level under fully funded pension is the lowest. Fully funded does not differ from private pension. The poorest people lacking income can neither save for retirement nor consume any goods. The MF system will improve social welfare as compared with fully funded, even though their economic growth rates coincide. Furthermore, a full compulsory pension by the MF system provides the highest welfare level within the MF system because the poorest people obtain consumption opportunities without risks. PG has its optimal social security tax rate. Its welfare level is always higher than that under full funding and is superior to MU with fringe benefits.

We also conducted quantitative analysis with plausible values of parameters to elucidate the qualitative results and to provide realistic examples. Results demonstrate that the growth rate under fully funded is the highest. With highly relative risk aversion, the welfare under PG overweighs those under MU (i.e., PG system with saving credit) and fully funded, irrespective of the types of social welfare functions and distribution of labor endowments. Furthermore, using the Benthamite welfare function, the optimal social security tax under PG increases with the population of low-income class. These results elucidate the effects of social security on economic growth and welfare through productivity and demographic shocks. The calculated tax rates complement those in the study by De Menil et al. (2016), although the setting differs. This complementary nature implies that the demographic structure (especially income class) strongly influences optimal social security tax rates.

Lastly, we consider the future directions for research in this area. The Rawlsian social welfare function provides a clear view of the redistributive policy. However, other criteria must be examined. In relation to the optimal social security, democratic determination of social security policy will provide new insights. Furthermore, incorporating endogenous fertility into the model proves to be interesting. Fundamentally, these extensions represent more realistic economic situations. Through our analyses, we can infer some conclusions of extensions. For example, social needs for redistribution will be weakened by the substitution of a political determination of social security tax rate for the Rawlsian welfare function. Such an extension can be expected to decrease the optimal social security tax rate and to increase economic growth rate. Therefore, our study provides an analytical basis of these extensive analyses.

Appendix

A. Saving functions under the funded pension systems

Equation (11). If the fully funded pension has risk-free return, then individuals prefer to purchase only the public pension. The first-order condition is

$$(w_t - \bar{s}_t)^{-\theta} = \frac{1}{1+\rho} E\left[\bar{R}_{t+1}^{1-\theta}\right] \bar{s}_t^{-\theta} \Leftrightarrow (w_t - \tau_t)^{-\theta} = \frac{\alpha^{1-\theta}}{1+\rho} \exp\left((1-\theta)\left(\mu_z + \frac{\sigma_z^2}{2}\right)\right) \tau_t^{-\theta}.$$

Solving this equation for saving yields

$$s_t = \frac{1}{1 + \beta \exp\left(-\left(\frac{1-\theta}{\theta}\right)\left(\mu_z + \frac{\sigma_z^2}{2}\right)\right)} w_t.$$

Using definitions for β, η, and A, we obtain Equation (10).

Equation (13). Inserting Equations (4) and (5) into the expected utility function yields

$$E[U_t] = \frac{(w_t h_t - s_t - \tau_t)^{1-\theta} - 1}{1-\theta} + \frac{1}{1+\rho} \frac{E\left[(R_{t+1} s_t + b_{t+1})^{1-\theta}\right] - 1}{1-\theta}.$$

When $\tau_t = b_{t+1} = 0$, the first-order condition of the utility maximization is

$$\frac{dE[U_t]}{ds_t} = -(w_t h_t - s_t)^{-\theta} + \frac{1}{1+\rho} E\left[R_{t+1}^{1-\theta}\right] s_t^{-\theta} = 0.$$

The first-order equation is rewritten as

$$(w_t h_t - s_t)^{-\theta} = \frac{\alpha^{1-\theta}}{1+\rho} \exp\left((1-\theta)\mu_x + \frac{(1-\theta)^2 \sigma_x^2}{2}\right) s_t^{-\theta}.$$

Solving the equation above with respect to s_t, we obtain

$$s_t = \frac{1}{1 + \beta A^{\eta} \lambda^{1-\theta}} w_t h_t.$$

With social security tax and benefit ($\tau_t > 0$ and $b_{t+1} > 0$), the same equation of saving must hold if

$$\tau_t \leq \frac{1}{1 + \beta A^{\eta} \lambda^{1-\theta}} w_t h_t. \tag{A1}$$

Therefore, Equation (13) is obtained if (A1) holds.

Equation (15). Under the MF pension, the first-order condition of the utility maximization is

$$\frac{dE[U_t]}{ds_t} = -[(1-\phi) w_t h_t - s_t]^{-\theta} + \frac{1}{1+\rho} E\left[R_{t+1}^{1-\theta} (s_t + \phi \bar{w}_t)^{-\theta}\right] = 0.$$

This equation becomes

$$[(1-\phi) w_t h_t - s_t]^{-\theta} = \frac{\alpha^{1-\theta}}{1+\rho} \exp\left((1-\theta)\mu_x + \frac{(1-\theta)^2 \sigma_x^2}{2}\right) (s_t + \phi \bar{w}_t)^{-\theta} = 0.$$

Solving the equation above for saving yields

$$s_t = \frac{(1-\phi)w_t h_t - \beta A^\eta \lambda^{1-\theta}\phi \bar{w}_t}{1+\beta A^\eta \lambda^{1-\theta}}.$$

Taking average of the individual saving function, we arrive at Equation (15).

B. Saving functions under unfunded pension systems

Equation (17). Using the budget equation of the pension system and Equation (3), we obtain $b_{t+1} = (1-\alpha)\psi X_{t+1}\bar{s}_t$. With this equation, the first-order condition is

$$\frac{dE[U_t]}{ds_t} = -[(1-\psi)w_t h_t - s_t]^{-\theta} + \frac{1}{1+\rho}E\big[R_{t+1}(R_{t+1}s_t + (1-\alpha)\psi X_{t+1}\bar{s}_t)^{-\theta}\big] = 0.$$

The equation above becomes

$$-[(1-\psi)w_t h_t - s_t]^{-\theta} + \frac{\alpha}{1+\rho}E\big[X_{t+1}^{1-\theta}\big](\alpha s_t + (1-\alpha)\psi \bar{s}_t)^{-\theta} = 0.$$

Therefore, we have

$$-[(1-\psi)w_t h_t - s_t] + \left(\frac{\alpha}{1+\rho}\right)^{-\frac{1}{\theta}} \exp\left(-\frac{(1-\theta)\mu_x}{\theta} - \frac{(1-\theta)^2\sigma_x^2}{2\theta}\right)[\alpha s_t + (1-\alpha)\psi \bar{s}_t] = 0.$$

Solving the equation above for individual saving leads to

$$s_t = \frac{(1-\psi)w_t h_t - \beta\chi\psi\lambda^{1-\theta}A^\eta \bar{s}_t}{1+\beta\lambda^{1-\theta}A^\eta}.$$

Using the individual saving, the average saving function is

$$\bar{s}_t = \left[\frac{1-\psi}{1+(1+\chi\psi)\beta\lambda^{1-\theta}A^\eta}\right]\bar{w}_t.$$

Equation (19). With fringe benefits (to give individuals saving incentive), the first-order condition of the utility maximization is

$$[(1-\psi)w_t h_t - s_t]^{-\theta} = \frac{1}{1+\rho}E\left[\frac{R_{t+1} + (1-\alpha)(1-\pi)\psi X_{t+1}}{\left(R_{t+1}s_t + \left[\pi + (1-\pi)\frac{s_t}{\bar{s}_t}\right](1-\alpha)\psi X_{t+1}\bar{s}_t\right)^\theta}\right].$$

This equation is rewritten as

$$[(1-\psi)w_t h_t - s_t]^{-\theta} = \frac{\alpha^{1-\theta}(1+(1-\pi)\chi\psi)A^{1-\theta}\lambda^{(\theta-1)\theta}\left(s_t + \left[\pi + (1-\pi)\frac{s_t}{\bar{s}_t}\right]\chi\psi\bar{s}_t\right)^{-\theta}}{1+\rho}.$$

Consequently, we obtain

$$s_t = \frac{(1-\psi)w_t h_t - (1+(1-\pi)\chi\psi)^{-\frac{1}{\theta}}\beta\pi\chi\psi A^\eta\lambda^{1-\theta}\bar{s}_t}{1+(1+(1-\pi)\chi\psi)^{\frac{\theta-1}{\theta}}\beta A^\eta\lambda^{1-\theta}} = \frac{(1-\psi)w_t h_t - \omega\beta\pi\chi\psi A^\eta\lambda^{1-\theta}\bar{s}_t}{1+(1+(1-\pi)\chi\psi)\omega\beta A^\eta\lambda^{1-\theta}}.$$

Using the individual saving function, the average saving function is

$$\bar{s}_t = \frac{(1-\psi)\bar{w}_t}{1 + (1 + (1-\pi)\chi\psi)^{-\frac{1}{\theta}}\beta A^{\eta}\lambda^{1-\theta}(1+\chi\psi)}.$$

C. Proof of Proposition 1

Lemmas 1 and 2 show $\gamma_{PG} \leq \gamma_{FF} = \gamma_{MF}$ for $\psi \geq 0$. Using Equations (13) and (20), we obtain

$$\gamma_{MU} = \gamma_{FF} \Leftrightarrow \hat{\psi} = \frac{(1-\widehat{\omega})\beta A^{\eta}\lambda^{1-\theta}}{1 + (1+\widehat{\omega}\chi)\beta A^{\eta}\lambda^{1-\theta}},$$

where $\widehat{\omega} \equiv \omega(\hat{\psi})$. Therefore, we have

$$\gamma_{MU} \gtreqless \gamma_{FF} \Leftrightarrow \frac{(1-\psi)(1-\alpha)A}{1 + (1+\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}} \gtreqless \frac{(1-\alpha)A}{1 + \beta A^{\eta}\lambda^{1-\theta}} \Leftrightarrow \psi \lesseqgtr \hat{\psi}.$$

Depending on θ, $\hat{\psi}$ can be larger or smaller than unity. $\gamma_{MU} = \gamma_{FF}$ ($\gamma_{MU} < \gamma_{FF}$) holds if $\psi = 0$ ($\psi = 1$). Consequently, a rise in ϕ must have a positive growth effect under the MU pension at $\phi = 0$. If θ is sufficiently small/(large), we obtain

$$\left.\frac{\partial \gamma_{MU}}{\partial \psi}\right|_{\psi=0} > 0 \left(\left.\frac{\partial \gamma_{MU}}{\partial \psi}\right|_{\psi=0} < 0\right).$$

Actually, $\gamma_{MU} > \gamma_{FF}$ holds if θ and ψ are sufficiently small. Furthermore, we have

$$\left.\frac{\partial \gamma_{MU}}{\partial \psi}\right|_{\psi=1} < 0.$$

The results obtained for the growth effect of ψ above indicate that $\hat{\psi}$ exists as $0 < \hat{\psi} < 1$ if θ is sufficiently small. Then, we arrive at $\gamma_{MU} > \gamma_{FF}$ for $0 < \psi < \hat{\psi}$ and a sufficiently small θ. Contrarily, if θ is sufficiently large, then $\hat{\psi} < 0$ holds. Consequently, $\gamma_{MU} < \gamma_{FF}$. We have $\gamma_{MU} > \gamma_{PG}$ because of

$$\frac{\partial \gamma_{MU}}{\partial \pi} < 0.$$

D. Proof of Proposition 4

One key determinant of social welfare level under PG is

$$\min U_t^{PG} = \frac{\left(\frac{\beta\chi\psi A^{\eta}\lambda^{1-\theta}\bar{s}_t}{1+\beta\lambda^{1-\theta}A^{\eta}}\right)^{1-\theta} - 1}{1-\theta} + \frac{1}{1+\rho}\frac{E\left[\left(\psi\bar{w}_{t+1} - \frac{\beta\chi\psi A^{\eta}\lambda^{1-\theta}\bar{s}_t}{1+\beta A^{\eta}\lambda^{1-\theta}}R_{t+1}\right)^{1-\theta}\right] - 1}{1-\theta}.$$

Using this equation, we obtain

$$\frac{\partial \min\left(\frac{U_t^{PG}}{w_t}\right)}{\partial\psi} = \left[\frac{\beta\chi A^\eta\lambda^{1-\theta}}{1+\beta\lambda^{1-\theta}A^\eta}\right]^{1-\theta}\left(\xi_{PG}+\psi\frac{\partial\xi_{PG}}{\partial\psi}\right)(\psi\xi_{PG})^{-\theta}$$

$$+\frac{1}{1+\rho}E\left[\left(\frac{c_{t+1}^o}{w_t}\right)^{-\theta}\left(1+\gamma_{PG}+\psi\frac{\partial\gamma_{PG}}{\partial\psi}-\frac{\beta\chi A^\eta\lambda^{1-\theta}\xi_{PG}}{1+\beta A^\eta\lambda^{1-\theta}}R_{t+1}-\frac{\beta\chi\psi A^\eta\lambda^{1-\theta}}{1+\beta A^\eta\lambda^{1-\theta}}\frac{\partial\xi_{PG}}{\partial\psi}R_{t+1}\right)\right].$$

For the corner values of ψ, the above partial derivative becomes

$$\left.\frac{\partial \min\left(\frac{U_t^{PG}}{w_t}\right)}{\partial\psi}\right|_{\psi=0} = \left[\frac{\beta\chi A^\eta\lambda^{1-\theta}}{1+\beta\lambda^{1-\theta}A^\eta}\right]^{1-\theta}\xi_{PG}^{1-\theta}\lim_{\psi\to0}\psi^{-\theta}+\frac{1}{1+\rho}E\left[\left(\frac{c_{t+1}^o}{w_t}\right)^{1-\theta}\right]=+\infty, \quad \text{(A2)}$$

$$\left.\frac{\partial \min\left(\frac{U_t^{PG}}{w_t}\right)}{\partial\psi}\right|_{\psi=1} = \left[\frac{\beta\chi A^\eta\lambda^{1-\theta}}{1+\beta\lambda^{1-\theta}A^\eta}\right]^{1-\theta}\lim_{\psi\to0}\frac{\partial\xi_{PG}}{\partial\psi}\lim_{\psi\to0}\xi_{PG}^{-\theta}$$

$$+\frac{1}{1+\rho}E\left[\left(\frac{c_{t+1}^o}{w_t}\right)^{-\theta}\left(\lim_{\psi\to0}\frac{\partial\gamma_{PG}}{\partial\psi}-\frac{\beta\chi A^\eta\lambda^{1-\theta}}{1+\beta A^\eta\lambda^{1-\theta}}\lim_{\psi\to0}\frac{\partial\xi_{PG}}{\partial\psi}R_{t+1}\right)\right]=-\infty.$$

These results indicate that there exists the value of $\tilde{\psi}$ satisfying

$$\left.\frac{\partial \min\left(\frac{U_t^{PG}}{w_t}\right)}{\partial\psi}\right|_{\psi=\tilde{\psi}}=0. \quad \text{(A3)}$$

Using (A2), (A3) and $\partial\gamma_{PG}/\partial\psi<0$ lead to

$$\left.\frac{\partial W_t^{PG}}{\partial\psi}\right|_{\psi=0}>0 \text{ and } \left.\frac{\partial W_t^{PG}}{\partial\psi}\right|_{\psi=\tilde{\psi}}<0.$$

Furthermore, there exists ψ^* satisfying $0<\psi^*<1$ and

$$\left.\frac{\partial W_t^{PG}}{\partial\psi}\right|_{\psi=\psi^*}=0.$$

If the pension is managed by the MU system, then the indirect utility function for generation t becomes

$$\min U_t^{MU}=\frac{\left[\frac{\omega\beta\pi\chi\psi A^\eta\lambda^{1-\theta}\bar{s}_t}{1+(1+(1-\pi)\chi\psi)\omega\beta A^\eta\lambda^{1-\theta}}\right]^{1-\theta}-1}{1-\theta}+\frac{1}{1+\rho}\frac{E\left[\left(\pi\psi\bar{w}_{t+1}\bar{h}_{t+1}\right)^{1-\theta}\right]-1}{1-\theta}.$$

The welfare effect of a rise in π includes its effect on the poorest people at generation t and its cumulative effect of income growth. Such a growth effect is captured as

$$\sum_{t=0}^{\infty}\left(\frac{1+\gamma_i}{1+\delta}\right)^t=\frac{1+\delta}{\delta-\gamma_i},$$

$$\frac{\partial}{\partial \pi}\left(\frac{1+\delta}{\delta-\gamma_{MU}}\right)=\frac{1+\delta}{(\delta-\gamma)^2}\frac{\partial \gamma_{MU}}{\partial \pi}<0.$$

With regard to the effect on the poorest people at generation t for $\pi = 0$, we have the following.

$$\lim_{\pi\to 0}\left[\frac{\partial \min\left(\frac{U_t^{MU}}{w_t}\right)}{\partial \pi}\right]=\frac{1}{1+\rho}\lim_{\pi\to 0}E\left[\psi^{1-\theta}[\pi(1+\gamma_{MU})]^{-\theta}(1+\gamma_{MU})\right]$$

$$+\lim_{\pi\to 0}\left[\frac{\omega\beta\pi\chi\psi A^{\eta}\lambda^{1-\theta}\xi_{MU}}{1+(1+(1-\pi)\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}}\right]^{-\theta}\left\{\frac{\omega\beta\chi\psi A^{\eta}\lambda^{1-\theta}\xi_{MU}\left[1+(1+\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}\right]}{[1+(1+\chi\psi)\omega\beta A^{\eta}\lambda^{1-\theta}]^2}\right\}$$

$$=+\infty,$$

This equation suggests that an increase in π improves social welfare because it dominates the negative welfare effect by decreasing the economic growth rate. Furthermore, we obtain the following equation for $\pi = 1$:

$$\lim_{\pi\to 1}\left[\frac{\partial \min\left(\frac{U_t^{MU}}{w_t}\right)}{\partial \pi}\right]$$

$$=\left[\frac{\beta\chi\psi A^{\eta}\lambda^{1-\theta}\xi_{MU}}{1+\beta A^{\eta}\lambda^{1-\theta}}\right]^{1-\theta}\left\{1+\left[\frac{1}{1+(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}}+\frac{(\theta-1)\beta A^{\eta}\lambda^{1-\theta}}{1+\beta A^{\eta}\lambda^{1-\theta}}\right]\frac{\chi\psi}{\theta}\right\}$$

$$+\frac{1}{1+\rho}\lim_{\pi\to 0}E\left[\psi^{1-\theta}[\pi(1+\gamma_{MU})]^{-\theta}\left(1+\gamma_{MU}+\pi\frac{\partial \gamma_{MU}}{\partial \pi}\right)\right]. \quad \text{(A4)}$$

The following equations hold:

$$\frac{1}{\omega}\frac{\partial \omega}{\partial \pi}=\frac{1}{\theta}\frac{\chi\psi}{1+(1-\pi)\chi\psi},\ \lim_{\pi\to 1}\left[\frac{1}{\omega}\frac{\partial \omega}{\partial \pi}\right]=\frac{\chi\psi}{\theta},\ \lim_{\pi\to 1}\omega=1,$$

$$\lim_{\pi\to 1}\left[\frac{1}{1+\gamma_{MU}}\frac{\partial(1+\gamma_{MU})}{\partial \pi}\right]=-\frac{(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}}{1+(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}}\frac{\chi\psi}{\theta}.$$

The first term on the right-hand side of Equation (A4) has an ambiguous sign, whereas the second term is negative.

$$\frac{1}{1+(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}}+\frac{(\theta-1)\beta A^{\eta}\lambda^{1-\theta}}{1+\beta A^{\eta}\lambda^{1-\theta}}\gtreqless 0 \Leftrightarrow \theta \gtreqless 1-\frac{1+\beta A^{\eta}\lambda^{1-\theta}}{[1+(1+\chi\psi)\beta A^{\eta}\lambda^{1-\theta}]\beta A^{\eta}\lambda^{1-\theta}}.$$

If θ is sufficiently large, then Equation (A4) becomes positive; the negative growth effect is decreased. Therefore, $\pi = 1$ is chosen. However, we obtain $0 < \pi < 1$ if θ is sufficiently small as Equation (A4) becomes negative.

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Tables

Table 1. Equilibrium growth rates within the funded system

System	Benchmark			Fully funded/Modified funded		
σ_x	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$
0	1.567	1.069	0.910	1.567	1.069	0.910
0.25	1.690	1.135	0.957	1.669	1.135	0.977
0.5	2.092	1.344	1.107	2.000	1.344	1.192
0.75	2.880	2.741	1.382	2.641	1.741	1.605
1	4.278	2.411	1.825	3.763	2.411	2.316

Table 2. Equilibrium growth rates within the unfunded system

System	Pay-as-you-go			Modified unfunded		
σ_x	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$
0	0.663	0.293	0.180	0.754	0.334	0.206
0.25	0.732	0.334	0.222	0.826	0.376	0.249
0.5	0.955	0.465	0.357	1.060	0.512	0.387
0.75	1.390	0.713	0.617	1.514	0.767	0.651
1	2.159	1.132	1.064	2.314	1.199	1.108

Table 3. Optimal tax rates and social welfare levels (Rawlsian welfare function)

Modified funded

	$\theta = 0.5$		$\theta = 1$		$\theta = 1.5$	
σ_x	ϕ^*	W^{MF}	ϕ^*	W^{MF}	ϕ^*	W^{MF}
0.00		25.551		4.053		3.916
0.25		27.825		4.261		3.941
0.50	100%	36.491	100%	4.886	100%	4.015
0.75		62.430		5.928		4.131
1.00		230.72		7.386		4.280

Pay-as-you-go

	$\theta = 0.5$		$\theta = 1$		$\theta = 1.5$	
σ_x	ψ^*	W^{PG}	ψ^*	W^{PG}	ψ^*	W^{PG}
0.00	14.5%	0.451		−4.669	22.1%	−3.270
0.25	14.0%	0.827		−4.461	22.0%	−3.178
0.50	12.3%	2.166	21.3%	−3.836	21.7%	−2.911
0.75	9.1%	5.538		−2.794	21.3%	−2.496
1.00	3.4%	18.982		−1.336	20.7%	−1.971

Modified unfunded

	$\theta = 0.5$			$\theta = 1$			$\theta = 1.5$		
σ_x	π^*	ψ^*	W^{MU}	π^*	ψ^*	W^{MU}	π^*	ψ^*	W^{MU}
0.00	86.3%	16.9%	0.470			−4.669		22.1%	−3.270
0.25	86.0%	16.4%	0.847			−4.461		22.0%	−3.178
0.50	85.0%	14.7%	2.190	100%	21.3%	−3.836	100%	21.7%	−2.911
0.75	82.9%	11.2%	5.571			−2.794		21.3%	−2.496
1.00	78.9%	4.4%	19.022			−1.336		20.7%	−1.971

Table 4. Optimal tax rates and social welfare levels (Benthamite welfare function)

Modified funded

	$\theta = 0.5$		$\theta = 1$		$\theta = 1.5$	
σ_z	ϕ^*	W^{MF}	ϕ^*	W^{MF}	ϕ^*	W^{MF}
0.50						
1.00						
3.00	100%	36.491	100%	4.886	100%	4.153
5.00						
10.0						

Pay-as-you-go

	$\theta = 0.5$		$\theta = 1$		$\theta = 1.5$	
σ_z	ψ^*	W^{PG}	ψ^*	W^{PG}	ψ^*	W^{PG}
0.50	0.0%	35.162	0.0%	4.469	0.0%	3.906
1.00	0.0%	31.420	0.0%	3.219	10.5%	3.246
3.00	3.0%	8.922	17.3%	−2.246	24.9%	0.778
5.00	10.9%	2.831	20.9%	−3.682	26.8%	0.060
10.0	12.3%	2.166	21.3%	−3.836	26.9%	−0.017

Modified unfunded

	$\theta = 0.5$			$\theta = 1$			$\theta = 1.5$		
σ_z	π^*	ψ^*	W^{MU}	π^*	ψ^*	W^{MU}	π^*	ψ^*	W^{MU}
0.50	0.0%	8.6%	37.018	3.4%	0.0%	4.469		0.0%	3.906
1.00	0.0%	8.6%	33.109	8.9%	0.0%	3.219		10.5%	3.246
3.00	29.5%	9.7%	9.327	100%	17.3%	−2.246	100%	24.9%	0.778
5.00	79.9%	13.8%	2.871	100%	20.9%	−3.682		26.8%	0.060
10.0	85.0%	14.7%	2.190	100%	21.3%	−3.836		26.9%	−0.017