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Pollution Externalities and Corrective Taxes in a Dynamic Small Open Economy

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Abstract

This study examines the effects of tax policies in a dynamic model of a polluted small open economy with two sources of flow pollution—consumption and production—controlled by consumption and income taxes. In this setting, accumulated pollution has a negative effect on households' utility. We show that in a decentralized dynamic competitive equilibrium under exogenous tax rates, whereas a permanent increase in consumption and income taxes unambiguously reduces the steady-state pollution stock, a temporary increase in these taxes may lead to more pollution in the long run. This outcome suggests that more stringent environmental policies might be ineffective if the regulation is only temporary. We also derive the socially optimal solution and examine the optimal tax paths to achieve the social optimum. If distaste and leisure effects are sufficiently strong, tax rates decrease along the optimal path as pollution increases over time, and vice versa.

Keywords: Pollution externalities, Small open economy, Permanent environmental policies, Temporary environmental policies, Optimal tax paths

JEL Classification Code: C61, F41, H23

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1 Introduction

Pollution externalities cause significant damage to the natural environment and a reduction of social welfare if there is no proper implementation of environmental policies. If pollution accumulates over time and pollution externalities are stock rather than flow ones, the issue becomes more complicated. As such, policymakers should consider the path of the pollution accumulation process in the economy and social welfare along which the economy and pollution evolve over time. In the absence of proper monitoring of the paths of pollution and economic variables, the government may implement too stringent or too lenient environmental policies compared to the socially optimal level. In either case, "government failure" could aggravate the market failure related to pollution externalities.

In this study, by examining the effects and design of an environmental policy in a dynamic small open economy with stock externalities caused by pollution, we highlight the issue of government failure in pollution control policies. Specifically, we consider an economy open to full international capital mobility, so that a domestic agent can borrow from and lend to the rest of the world freely at an exogenously given world interest rate. We consider two types of pollution by source: one from firms' production activities and the other from households' consumption activities. To control emissions in the respective economic sectors, this study assumes that a national government implements tax policies to control pollution.

We assume that taxes are set based on the source of pollution. Specifically, we consider an income tax imposed on production-generated pollution and a consumption tax on consumption-generated pollution.[1] We analyze the effects of temporary policy changes (in which the government changes the tax rates from the original levels for a certain period of time and then returns to the original levels) as well as permanent policy changes. We find that, whereas a permanent increase in tax rates (i.e., more stringent environmental policy) reduces the steady-state level of pollution stock, a temporary increase in tax rates may lead to more pollution stock in the long run. In other words, in a small open economy, government failure may occur in the sense that more stringent environmental regulation may be ineffective if the regulation is only temporary.

There is anecdotal evidence that an environmental protection scheme can be invalidated before it takes off, resulting in worse environmental quality than even before the implementation of the environmental policy. This can occur because governments and policymakers are relatively myopic compared to the *long-lived environment* (John et al., 1995) and are affected by politics. For example, in Australia, a carbon-pricing scheme was introduced by the Gillard government

[1] We do not consider pollution abatement activities. Therefore, increasing consumption taxes directly decreases consumption, thereby controlling the pollution caused by consumption activities. Similarly, an increase in income taxes leads to less labor supply, and thus, reduces the pollution caused by production activities.

in 2011 through the Clean Energy Act 2011, which came into effect in July 2012. However, the scheme was repealed in July 2014 by the Abbott government after a change of administration.[2] According to recent estimates by carbon consultancy NDEVR Environmental, greenhouse gas (GHG) emissions for the first quarter of the fiscal year 2018 were the second highest quarterly results in 5 years.[3] The estimates also show a steady annual increase in GHG emissions, and thus, emissions may outweigh those before the implementation of the carbon tax. Therefore, there are significant policy implications of government failure caused by the implementation of environmental policies that are only temporary.

In addition to investigating the effects of policy changes for controlling pollution in a decentralized economy, we examine the socially optimal path and tax rules to achieve the social optimum. We show that households' preferences play a key role in the characteristics of the socially optimal path. Specifically, assuming that the pollution stock has a distaste effect on consumption (as in Michel and Rotillon, 1995) and a leisure effect on labor supply (as in Bosi et al., 2015, and Heijdra et al., 2015), we can conclude that, when these effects have strong impacts, the optimal rates of consumption and income tax decrease as the pollution stock increases. This finding contrasts with the conventional belief that it is optimal for an economy to implement more stringent environmental policies when environmental problems become worse. On the other hand, if the abovementioned effects are not strong enough, such a paradoxical result does not occur; we show that more stringent environmental policies should be implemented in response to larger pollution along an optimal path.

Our study is related to two specific research areas: dynamic models with pollution and dynamic international macroeconomics. In the dynamic analysis of an economy with pollution, several studies incorporate pollution into economic growth models. In most existing studies, the source of pollution is either production or consumption. For instance, Lopez (1994) and Selden and Song (1995) construct dynamic models with pollution generated by production, while John and Pecchenino (1994), John et al. (1995), and McConnell (1997) develop dynamic models with pollution generated by consumption. These studies have a common research interest, namely, the possibility of the environmental Kuznets curve hypothesis indicating an inverted U-shaped relationship between economic development and environmental degradation. Other research interests in the literature on economic growth and pollution include examination of whether sustainable growth can be achieved by using endogenous growth models with pollution (e.g.,

[2]A more recent example involves the U.S. energy and environmental policies of the Trump administration, which have been characterized by the systematic repeal of measures by the preceding Obama administration. This policy change has triggered significant concerns about future environmental damage (see, e.g., *The Guardian*, October 3, 2018).

[3]http://ndevr.com.au/environmental/tracking-2-degrees-fy18-q1.

Huang and Cai, 1994; Bovenberg and Smulders, 1995; Michel and Rotillon, 1995; Smulders and Gradus, 1996; Chevé, 2000; Greiner, 2005; Gupta and Barman, 2009). Although we use a dynamic macroeconomic model with pollution, as the abovementioned studies do, our research interest is significantly different from theirs. We newly compare the effects of the temporary and permanent implementation of pollution control policies, which is a novel contribution of this research.

Dynamic models of pollution have also analyzed the design of optimal tax policies to achieve a socially optimal dynamic path (e.g., Tahvonen and Kuuluvainen, 1991, 1993; Bovenberg and Smulders, 1995; Fullerton and Kim, 2008; Aloi and Tournemaine, 2011; Wang et al., 2015).[4] All these studies are confined to closed economy models and, more importantly, merely propose optimal tax formulas to achieve the first-best outcome. By contrast, we carry out a more detailed analysis of the dynamic paths of the optimal tax rates.

In terms of model structure, our study is an extension of the dynamic international macroeconomic models analyzed by Sen and Turnovsky (1990), Turnovsky (1997), and Schubert and Turnovsky (2002), who examine the effects of temporary policies in small open economies. These studies find that, when a policy temporarily changes under the assumption of perfect foresight, the small open economy does not return to its original steady-state equilibrium after the policy variables return to their original levels. This finding contrasts with that obtained under the assumption of a closed economy, for which the temporary implementation of public policies has no impact on the long-run equilibrium, that is, the long-run equilibrium coincides with the original steady state. Note that existing studies on this issue do not consider pollution problems, and thus, do not analyze the effects of temporary environmental policies.

The motivation for our study is closely related to that of Nakamoto and Futagami (2016) in that we cast doubt on the effectiveness of environmental protection policies in open economies. Nakamoto and Futagami (2016) consider a small open economy with renewable natural resources, such as forestry, fish, and wildlife stocks, and assume that private agents control the harvest and resource recovery rates subject to the dynamics of natural resources. In their model, resource goods are exported to the rest of the world, and thus, the source of income is in the small open economy. By contrast, in our model, the pollution stock affects households' utility as pure externalities, and thus, we exclude the dynamics of pollution accumulation in private agents' optimization problems. Moreover, although Nakamoto and Futagami (2016) consider tax policies to control natural resources, they do not derive tax rates to achieve the social optimum, while we derive optimal tax rules that mimic the socially optimal path in the economy.

[4] Bovenberg and Heijdra (1998) study the effects of environmental taxation using an overlapping generations model.

The remainder of this paper is organized as follows. In the next section, we present our basic model. Section 3 derives the dynamic equilibrium of our small open economy and shows the uniqueness of the steady-state equilibrium with saddle-path stability. Section 4 examines the effects of environmental policies. Section 5 analyzes the optimal dynamic path of the economy, determined by a social planner, and derives the optimal tax policies that mimic the socially optimal path in a decentralized economy. Section 6 conducts numerical simulations. Section 7 concludes.

2 Baseline model

We consider a small open economy facing a constant world interest rate, denoted by r. The population in this economy is assumed to be constant and normalized to unity. Denoting the time index by t, we express the production function that satisfies constant returns to scale with respect to capital ($\hat{k}_t$) and labor (l_t) as

$$y_t = F(\hat{k}_t, l_t) = l_t f(k_t),$$

where y_t is the output and $k_t \equiv \hat{k}_t / l_t$ denotes capital intensity. The intensive form of the production function, $f(k_t)$, is monotonically increasing, strictly concave in k_t, and satisfies the Inada conditions. Considering competitive markets for factors and final goods, the real rental and wage rates, r and w_t, respectively, are determined by

$$r = f'(k_t), \quad w_t = f(k_t) - k_t f'(k_t). \tag{1a}$$

Since r is constant for a small open economy, it follows from (1a) that capital intensity is constant, and hence, the wage rate is fixed as well: $k_t = \bar{k}$ and $w_t = \bar{w}$. As a result, the economy's output is simply a linear function of labor input, given by

$$y_t = l_t f(\bar{k}). \tag{1b}$$

2.1 Pollution

We assume that the pollution flow caused by consumption and output accumulates over time, but a fraction θ of the current pollution stock reduces due to the assimilative capacity of the environment. Then, the dynamics of pollution accumulation is given by

$$\dot{P}_t = \alpha_c G(c_t) + \alpha_y N(l_t f(\bar{k})) - \theta P_t, \tag{2}$$

where P_t denotes the aggregate pollution stock, c_t the level of consumption, and θ the natural decay rate of the pollution stock. The initial stock of aggregate pollution, P_0, is exogenously given. Functions $G(c_t)$ and $N(l_t f(\bar{k}))$ represent the flow levels of pollution emissions caused by consumption and output, respectively. We assume the following:

$$G'(c_t) > 0,\ G''(c_t) \geq 0,\ N'(l_t f(\bar{k})) > 0,\ N''(l_t f(\bar{k})) \geq 0, \tag{3a}$$

$$\lim_{c_t \to 0} G(c_t) = \lim_{c_t \to 0} G'(c_t) = 0,\ \lim_{l_t \to 0} N(l_t f(\bar{k})) = \lim_{l_t \to 0} N'(l_t f(\bar{k})) = 0. \tag{3b}$$

These conditions mean that the more active the economy is, the higher is the pollution flow, and the relationship is convex. We assume that emission coefficients α_c and α_y are non-negative. If both α_c and α_y are positive, we consider pollutants emitted by both consumption and production activities. Examples of such pollutants include carbon dioxide (which almost all economic activities emit) and nitrogen oxides (which are emitted by thermal power plants and households' gasoline-driven cars). There are also cases in which only consumption activities generate pollution ($\alpha_c > 0 = \alpha_y$) or only production activities generate pollution ($\alpha_c = 0 < \alpha_y$). An example of the former is household waste and an example of the latter is industry effluent.

2.2 Households

In this economy, households derive utility from consumption c_t and disutility from labor supply l_t. In addition, owing to pollution externalities, households suffer from the aggregate pollution stock P_t. We consider the situation in which the pollution stock affects the marginal utility of consumption and marginal disutility of labor supply separately by incorporating two non-separable utility functions $u(c_t, P_t)$ and $\omega(l_t, P_t)$.[5] Then, a representative household's lifetime utility is given by

$$U_0 \equiv \int_0^\infty \big\{u(c_t, P_t) - \omega(l_t, P_t)\big\} e^{-\rho t} dt, \tag{4}$$

where $\rho(> 0)$ is the rate of time preference. Both $u(\cdot,\cdot)$ and $\omega(\cdot,\cdot)$ are assumed to be twice continuously differentiable and have the following partial derivatives:

$$u_c(c_t, P_t) > 0,\ u_{cc}(c_t, P_t) < 0,\ u_P(c_t, P_t) < 0,\ u_{PP}(c_t, P_t) < 0, \tag{5a}$$

$$\omega_l(l_t, P_t) > 0,\ \omega_{ll}(l_t, P_t) > 0,\ \omega_P(l_t, P_t) > 0,\ \omega_{PP}(l_t, P_t) > 0. \tag{5b}$$

In summary, the utility function $u(c_t, P_t)$ is increasing and strictly concave in the level of consumption, and decreasing and concave in the level of pollution stock. The disutility function

[5] In other words, when considering a more general utility function, $U(c_t, l_t, P_t)$, we assume that $U_{cl} = 0$. More importantly, even if $U_{cl} \neq 0$, the main results do not change provided the following additional conditions hold:

$$U_{ll}(c_t, l_t, P_t) U_{cc}(c_t, l_t, P_t) \geq U_{cl}(c_t, l_t, P_t)^2 \quad \text{and} \quad \frac{U_{lP}(c_t, l_t, P_t) U_{cc}(c_t, l_t, P_t)}{U_{cl}(c_t, l_t, P_t) U_{cP}(c_t, l_t, P_t)} \geq 1 \geq \frac{U_{lP}(c_t, l_t, P_t) U_{cc}(c_t, l_t, P_t)}{U_{cP}(c_t, l_t, P_t) U_{ll}(c_t, l_t, P_t)}.$$

These conditions guarantee the existence and saddle-point stability of a steady state.

$\omega(l_t, P_t)$ is strictly decreasing and convex in l_t and P_t. We also assume the Inada conditions as follows:

$$\lim_{l_t \to 0} \omega_l(l_t, P_t) = 0, \ \lim_{l_t \to \infty} \omega_l(l_t, P_t) = \infty, \ \lim_{c_t \to 0} u_c(c_t, P_t) = \infty, \ \lim_{c_t \to \infty} u_c(c_t, P_t) = 0. \tag{5c}$$

Regarding cross-derivatives, several studies assume separability of the utility function in the consumption and pollution stock (i.e., $u_{cP} = 0$ and $\omega_{lP} = 0$), while others make other assumptions. In particular, when the cross-derivative between consumption and pollution is negative, that is, $u_{cP}(c_t, P_t) < 0$, the marginal utility of consumption decreases with the pollution stock. Then, pollution has a distaste effect on consumption, as per Michel and Rotillon (1995). On the other hand, the interplay between pollution and labor supply remains theoretically ambiguous; however, as in Bosi et al. (2015), recent empirical contributions have shown a significant negative impact of pollution on labor supply, which is called the leisure effect. One reason is that pollution may worsen working conditions and provide workers with an incentive to substitute leisure for working time. Therefore, in light of the distaste and leisure effects, we make the following assumptions:

$$u_{cP}(c_t, P_t) < 0, \quad \omega_{lP}(l_t, P_t) > 0. \tag{6}$$

For example, the following pair of functions satisfies the abovementioned assumptions (5a)–(5c) and (6):

$$u(c_t, P_t) = v(c_t)H(P_t) = \left(\frac{c_t^{1-\gamma}}{1-\gamma}\right)\left(\bar{h} - P_t^{1+\eta}\right), \quad \omega(l_t, P_t) = \frac{(l_t^{\delta} P_t^{\beta})^{1+\epsilon}}{1+\epsilon}. \tag{7}$$

We assume that $1 > \gamma > 0$, so that $v(c_t)$ always has a positive value. Furthermore, we assume $\bar{h}$ is a positive parameter that satisfies $\bar{h}^{1/(1+\eta)} > P_t$, and function $H(P_t)$ always has a positive value, where we assume that $\eta > 0$. Then, we can observe that $u_c(c_t, P_t) = v'(c_t)H(P_t) > 0$, $u_{cc}(c_t, P_t) = v''(c_t)H(P_t) < 0$, $u_{cP}(c_t, P_t) = v'(c_t)H'(P_t) < 0$, $u_P(c_t, P_t) = v(c_t)H'(P_t) < 0$, and $u_{PP}(c_t, P_t) = v(c_t)H''(P_t) < 0$. As for the disutility function in $\omega(l_t, P_t)$, we can easily confirm that $\omega(l_t, P_t)$ in (7) satisfies the assumptions in (5b), (5c), and (6), where we assume that $\delta > 0$, $\beta > 0$, $\epsilon > -1$, $\delta(1+\epsilon) > 1$, and $\beta(1+\epsilon) > 1$.

The accumulation of foreign asset holdings, b_t, evolves as

$$\dot{b}_t = rb_t + (1-\tau^y)l_t f(\bar{k}) - (1+\tau^c)c_t + z_t, \tag{8}$$

where z_t is the lump-sum transfer, τ^y the rate of income tax, and τ^c the rate of consumption tax. We assume that $\tau^c > -1$ and $\tau^y < 1$. [6] In particular, when $\tau^j < 0$ for $j = y, c$, the government

[6]In the analysis of optimal taxation in Section 5, we allow for a case in which τ^c and τ^y are not necessarily positive.

implements a subsidy policy, which leads to more pollution. Moreover, in the baseline model, we assume that τ^y and τ^c are fixed.

The government keeps the following balanced budget over time:

$$\tau^c c_t + \tau^y l_t f(\bar{k}) = z_t. \tag{9}$$

3 Dynamic equilibrium in a decentralized small open economy

3.1 Dynamic system in a decentralized economy

In a decentralized market economy, disutility from the pollution stock takes the form of external diseconomies. Therefore, a representative household maximizes its lifelong utility (4) subject to flow budget constraint (8), without considering the evolution of pollution (2). We set the current-value Hamiltonian as follows:

$$\mathcal{H} = u(c_t, P_t) - \omega(l_t, P_t) + \lambda_t \left(rb_t + (1-\tau^y) l_t f(\bar{k}) - (1+\tau^c) c_t + z_t \right), \tag{10}$$

where λ_t shows the shadow value associated with (8).

The first-order necessary conditions are

$$H_{c_t}: \; u_c(c_t, P_t) = \lambda_t (1+\tau^c), \tag{11a}$$

$$H_{l_t}: \; \omega_l(l_t, P_t) = \lambda_t (1-\tau^y) f(\bar{k}), \tag{11b}$$

$$H_{b_t}: \; \lambda_t r = -\dot{\lambda}_t + \rho \lambda_t. \tag{11c}$$

The transversality condition is given by

$$\lim_{t \to \infty} \lambda_t b_t e^{-\rho t} = 0. \tag{11d}$$

In the following, since the rate of time preference ρ and the interest rate r are both fixed, we require $r = \rho$ for our system to have a finite interior steady-state value for the shadow value of foreign assets. Therefore, assuming that $\rho = r$, (11c) yields the time-invariant level of shadow value λ_t:

$$\lambda_t = \bar{\lambda}. \tag{12}$$

Note that the value of $\bar{\lambda}$ is endogenously determined, so that it is consistent with equilibrium conditions, which is discussed later.

Substituting (12) into (11a), we can obtain the household's optimal consumption as follows:

$$c_t = c(P_t, \bar{\lambda}, \tau^c), \tag{13}$$

which has the following properties:

$$\frac{\partial c_t}{\partial P_t} = -\frac{u_{cP}}{u_{cc}}(<0), \quad \frac{\partial c_t}{\partial \bar{\lambda}} = \frac{u_c}{\bar{\lambda} u_{cc}}(<0), \quad \frac{\partial c_t}{\partial \tau^c} = \frac{u_c}{(1+\tau^c)u_{cc}}(<0).$$

Moreover, from (6), we can observe that

$$\infty \geq \lim_{P_t \to 0} c(P_t, \bar{\lambda}, \tau^c) = c(0, \bar{\lambda}, \tau^c) > \lim_{P_t \to \infty} c(P_t, \bar{\lambda}, \tau^c) = c(\infty, \bar{\lambda}, \tau^c) \geq 0. \tag{14}$$

Similarly, by substituting (12) into (11b), we can obtain the household's optimal labor supply as follows:

$$l_t = l(P_t, \bar{\lambda}, \tau^y), \tag{15}$$

which satisfies

$$\frac{\partial l_t}{\partial P_t} = -\frac{\omega_{lP}}{\omega_{ll}}(<0), \quad \frac{\partial l_t}{\partial \bar{\lambda}} = \frac{\omega_l}{\bar{\lambda}\omega_{ll}}(>0), \quad \frac{\partial l_t}{\partial \tau^y} = -\frac{\omega_l}{(1-\tau^y)\omega_{ll}}(<0).$$

The assumption of cross-derivatives in (6) leads to the following:

$$\infty \geq \lim_{P_t \to \infty} l(P_t, \bar{\lambda}, \tau^y) = l(\infty, \bar{\lambda}, \tau^y) > \lim_{P_t \to 0} l(P_t, \bar{\lambda}, \tau^y) = l(0, \bar{\lambda}, \tau^y) \geq 0. \tag{16}$$

Substituting (13) and (15) into (2), we obtain the following differential equation, which represents the dynamics of pollution stock accumulation along the equilibrium path:

$$\dot{P}_t = \alpha_c G(c(P_t, \bar{\lambda}, \tau^c)) + \alpha_y N(l(P_t, \bar{\lambda}, \tau^y) f(\bar{k})) - \theta P_t. \tag{17}$$

3.2 Steady-state equilibrium and stability

Let us denote the steady-state value of each variable by an asterisk. Then, for a given value for $\bar{\lambda}$, the aggregate level of the pollution stock in the steady state, P^*, is determined by $\dot{P}_t = 0$ in (17) as follows:

$$\Psi(P^*) \equiv \alpha_c G(c(P^*, \bar{\lambda}, \tau^c)) + \alpha_y N(l(P^*, \bar{\lambda}, \tau^y) f(\bar{k})) - \theta P^* = 0. \tag{18}$$

Using (14) and (16), we can show that the function $\Psi(\cdot)$ has the following properties:

$$\lim_{P \to 0} \Psi(P) = \alpha_c G(c(0, \bar{\lambda}, \tau^c)) + \alpha_y N(l(0, \bar{\lambda}, \tau^y) f(\bar{k})) > 0,$$
$$\lim_{P \to \infty} \Psi(P) = \alpha_c G(c(\infty, \bar{\lambda}, \tau^c)) + \alpha_y N(l(\infty, \bar{\lambda}, \tau^y) f(\bar{k})) - \theta \times \infty = -\infty.$$

We can also show that since c and l are decreasing in P (see (13) and (15)), $\Psi(\cdot)$ is a decreasing function of P:

$$\Psi'(P) = \alpha_c G'(c(P, \bar{\lambda}, \tau^c)) \frac{\partial c}{\partial P} + \alpha_y N'(l(P, \bar{\lambda}, \tau^y) f(\bar{k})) f(\bar{k}) \frac{\partial l}{\partial P} - \theta(<0).$$

Thus, there is a unique solution P^* that satisfies (18), given the level of shadow value $\bar{\lambda}$.

Let us denote the stable root toward the steady state in (17) by μ. In light of (17) and the uniqueness of P^*, μ is uniquely determined as follows:

$$\mu = \frac{\partial \dot{P}_t}{\partial P_t} = \alpha_c G'(c(P^*, \bar{\lambda}, \tau^c))\frac{\partial c^*}{\partial P^*} + \alpha_y N'(l(P^*, \bar{\lambda}, \tau^y) f(\bar{k})) f(\bar{k}) \frac{\partial l^*}{\partial P^*} - \theta (< 0). \tag{19}$$

These results are summarized by the following proposition.

Proposition 1 *The steady-state equilibrium is uniquely determined and satisfies saddle-path stability.*

Given the initial level of the pollution stock, P_0, and its steady-state level, P^*, which satisfies (18), the linearly approximated dynamics of the pollution stock are as follows:

$$P_t = P^* + (P_0 - P^*)e^{\mu t}, \tag{20}$$

where μ is given by (19).

In light of (13) and (15), the linear approximation of (8) yields

$$\dot{b}_t = r(b_t - b^*) + \left(f(\bar{k})\frac{\partial l_t}{\partial P_t} - \frac{\partial c_t}{\partial P_t} \right)(P_t - P^*), \tag{21}$$

where the steady-state level of foreign asset b^* must satisfy, as per (8) and (9),

$$rb^* + f(\bar{k})l(P^*, \bar{\lambda}, \tau^y) = c(P^*, \bar{\lambda}, \tau^c). \tag{22}$$

By solving (21) and substituting (20) into it, we obtain

$$b_t = b^* - \frac{P^* - P_0}{\rho - \mu} \Bigg(\underbrace{\frac{\partial c_t}{\partial P_t} - f(\bar{k})\frac{\partial l_t}{\partial P_t}}_{(\#1)} \Bigg) e^{\mu t}. \tag{23}$$

The sign of (#1) in (23) may be positive or negative. For instance, when $(0 >) f(\bar{k})\frac{\partial l_t}{\partial P_t} > \frac{\partial c_t}{\partial P_t}$ (i.e., the negative impact of the pollution stock on consumption is larger than its negative impact on the labor supply), (#1) is negative. Given that $\frac{\partial c_t}{\partial P_t}$ and $\frac{\partial l_t}{\partial P_t}$ depend on cross-derivatives u_{cP} and ω_{lP} in (13) and (15), respectively, we argue that this case occurs when the distaste effect of the pollution stock on consumption is sufficiently large. Since households do not want to decrease consumption to a large degree, they reduce their savings, and thus, foreign assets. In other words, since future investment is substituted for current consumption, foreign assets decrease over time. When $f(\bar{k})\frac{\partial l_t}{\partial P_t} < \frac{\partial c_t}{\partial P_t}(< 0)$, the opposite occurs, that is, households' foreign assets increase over time. Note that, at time $t = 0$, (23) can be rewritten as

$$b^* - b_0 = \frac{P^* - P_0}{\rho - \mu} \left(\frac{\partial c_t}{\partial P_t} - f(\bar{k})\frac{\partial l_t}{\partial P_t} \right). \tag{24}$$

The dynamic behavior of the pollution stock and that of foreign assets is illustrated in Figure 1, in which we define the pollution flow as follows:

$$\Omega_t \equiv \alpha_c G(c(P_t, \bar{\lambda}, \tau^c)) + \alpha_y N(l(P_t, \bar{\lambda}, \tau^y) f(\bar{k})).$$

There are two possible relationships between P_t and b_t, which reflect the fact that the sign of (#1) in (23) can be either positive or negative, depending on the relative magnitude of distaste effect $u_{cP} < 0$ and leisure effect $\omega_{lP} > 0$. If the distaste effect is sufficiently large such that (#1) is negative, then the foreign assets are negatively related to the pollution stock, as illustrated in Figure 1(a). By contrast, if the distaste effect is relatively weak, P_t and b_t are positively related, as shown in Figure 1(b).

[Figure 1 around here.]

Consider the case in which (#1) is negative, corresponding to Figure 1(a). Suppose that the initial pollution stock, P_0, satisfies $\Omega_0 > \theta P_0$. In this case, the pollution flow is initially greater than its natural decay level, and thus, there is larger pollution stock subsequently. As shown in the upper panel of Figure 1(a), this trend will continue until the economy reaches the steady state E, where $\Omega^* = \theta P^*$ holds. Along the transition path from P_0 to P^*, (23) indicates that foreign assets decrease over time from b_0 to b^* if (#1) is negative. This is shown in the lower panel of Figure 1(a). The dynamics of the aggregate pollution stock and foreign assets in the case in which (#1) is positive, as illustrated in Figure 1(b), can be interpreted analogously.

In summary, for a given pair of taxes (τ^c, τ^y), the steady-state solutions for b, P, and associated shadow price $\bar{\lambda}$ are determined by (18), (22), and (24).

4 Long-run effects of tax policies on pollution: Permanent and temporary policy changes

Having derived the dynamic equilibrium of our small open economy, we can now examine the effects of tax policies on the aggregate pollution stock in the long run. We consider temporary policy changes, as well as permanent ones. In closed economy models, a temporary policy change affects the transitional path over time, and after the temporary change is removed, the economy gradually returns to the original steady state. Thus, we conclude that the temporary policy change does not have any qualitative impacts on the long-run economy. By contrast, in small open economy models, when a policy is implemented temporarily, the long-run steady state does not coincide with the original one (see, e.g., Turnovsky, 1997). If the steady state after the temporary implementation of a more stringent environmental policy has more pollution than what the original policy achieves, this is quite different from our expectation, and thus, we should be more careful about the efficacy of environmental policies in an open economy.

To discuss the effects of permanent and temporary changes in taxes to control pollution, we assume that the economy is initially at the steady state, in which the pair of tax rates is denoted by (τ_0^c, τ_0^y). We denote the steady-state values of pollution stock and foreign assets corresponding to these tax rates by P_0^* and b_0^*, respectively, and the shadow price by $\bar{\lambda}_0$. We represent these steady-state equilibrium values as follows:[7]

$$P_0 = P_0^* = P(\bar{\lambda}_0, \tau_0^c, \tau_0^y), \quad P_{\bar{\lambda}} < (>) 0 \text{ if } \alpha_c > 0 \text{ and } \alpha_y = 0 \ (\alpha_c = 0 \text{ and } \alpha_y > 0),$$

$$P_{\tau^c} < (=) 0 \text{ if } \alpha_c > 0 \ (\alpha_c = 0), \quad P_{\tau^y} < (=) 0, \text{ if } \alpha_y > 0 \ (\alpha_y = 0), \tag{25a}$$

$$b_0 = b_0^* = B(\bar{\lambda}_0, \tau_0^c, \tau_0^y), \quad B_{\tau^c} < 0, \quad B_{\tau^y} > 0, \tag{25b}$$

$$\bar{\lambda}_0 = L(\tau_0^c, \tau_0^y), \quad L_{\tau^c} < 0, \quad L_{\tau^y} > 0. \tag{25c}$$

4.1 Permanent policy changes

We begin with a permanent increase in the rate of consumption and/or income tax, so that the national government sets higher tax rates than the initial level(s) τ_0^c and/or τ_0^y from then on. The long-run consequence of such a permanent environmental policy on the pollution stock is as follows.

Proposition 2 *A permanent increase in the rate of the respective tax (i.e., consumption or income) leads to a reduction in the steady-state level of the pollution stock.*

Proof. See Appendix B. ■

When the government increases each tax permanently, the long-run pollution stock is smaller than its original steady-state level. More importantly, this finding is robust to the type of taxes and source of pollution. We begin with the case in which both consumption and production activities generate pollution (i.e., $\alpha_c > 0$ and $\alpha_y > 0$). Assume a permanent increase in consumption tax rate τ^c, with income tax rate τ^y remaining unchanged. This has a direct effect on the steady-state pollution stock, as represented by (25a); the higher the consumption tax rate is, the less the economy consumes, which results in less pollution from consumption in the long run. Note also that in light of (25a) and (25c), an increase in the consumption tax rate has an indirect effect on the steady-state pollution stock via a change in the shadow value $\bar{\lambda}$. However, this indirect effect is shown to be dominated by the direct effect of the increase in τ^c.[8] Therefore, we conclude that a permanent increase in the consumption tax rate unambiguously reduces pollution in the long run. The effect of an increase in the income tax rate τ^y can be similarly explained.

[7] See Appendix A.

[8] See also (26a), (26b), and Appendix B.

We next consider a special case in which pollution is generated only from consumption ($\alpha_c > 0 = \alpha_y$), and, in addition, assume that the government increases only income tax rate τ_y permanently. In this case, (25a) indicates that the increase in the income tax rate does not have a direct impact on the pollution stock, because the pollution flow is generated only by consumption activities. There is only an indirect effect from a change in the shadow value of future consumption (i.e., accumulation of foreign assets), $\bar{\lambda}$. Specifically, from (25c), an increase in τ^y has a positive effect on $\bar{\lambda}$. In other words, since current consumption is substituted by future consumption, current consumption decreases, which reduces the current pollution flow. Since only this indirect effect remains in the long run, the steady-state pollution stock decreases as well.

Figure 2 illustrates the abovementioned long-run effects of an increase in tax rates on the pollution stock and foreign assets. Suppose that the economy is initially at E_0 and consider an increase in τ^c or τ^y. Then, given the pollution stock, the Ω_t curve shifts as follows:[9]

$$\left.\frac{\partial \Omega_t}{\partial \tau^c}\right|_{P_t=\text{const}} = \underbrace{\alpha_c G'(c^*)\frac{\partial c^*}{\partial \tau^c}}_{(-)} + \underbrace{\left(\alpha_c G'(c^*)\frac{\partial c^*}{\partial \bar{\lambda}} + \alpha_y N'(l^* f(\bar{k})) f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right)}_{(+)\text{ or }(-)} \underbrace{L_{\tau^c}}_{(-)} < 0, \tag{26a}$$

$$\left.\frac{\partial \Omega_t}{\partial \tau^y}\right|_{P_t=\text{const}} = \underbrace{\alpha_y N'(l^* f(\bar{k})) f(\bar{k})\frac{\partial l^*}{\partial \tau^y}}_{(-)} + \underbrace{\left(\alpha_c G'(c^*)\frac{\partial c^*}{\partial \bar{\lambda}} + \alpha_y N'(l^* f(\bar{k})) f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right)}_{(+)\text{ or }(-)} \underbrace{L_{\tau^y}}_{(+)} < 0. \tag{26b}$$

Therefore, as shown in the upper panels of Figures 2(a) and 2(b), an increase in the respective taxes shifts the Ω_t curve downward. Since the θP_t line is not affected by changes in tax rates, the new steady state is denoted by E_1, where the pollution stock is smaller than the initial steady-state level: $P_1^* < P_0^*$. The effect on the long-run level of foreign assets depends on the sign of (#1) in (23): if (#1) has a negative (positive) sign, an increase in tax rates increases (reduces) the long-run level of foreign assets, as shown in the lower panel of Figure 2(a) (2(b)).

[Figure 2 around here.]

4.2 Temporary policy changes

We here consider the case in which the government announces a temporary change in the taxes from the original levels (τ_0^c, τ_0^y) to the new levels (τ_1^c or τ_1^y). After the initial change of taxes at time 0, the new tax rates are kept unchanged for $t = [0, T)$; they then return to the original levels at time T and remain there (i.e., for $t = [T, \infty]$). In this case, the unanticipated change leads to an initial jump of shadow value $\bar{\lambda}$. More importantly, because of the assumption of

[9]See Appendix B.

perfect foresight, households can initially anticipate that the tax rate returns to the original level at time T. This means that, except for the initial jump at time 0, the shadow value $\bar{\lambda}$ does not change over time, including time T. In other words, the shadow value is kept fixed to sustain the intertemporal solvency condition. Since only this effect remains in the long run, the temporary policy changes have long-run impacts on pollution.

Let us define

$$\Lambda \equiv \alpha_c G'(c^*) \frac{\partial c^*}{\partial \bar{\lambda}} + \alpha_y N'(l^* f(\bar{k})) f(\bar{k}) \frac{\partial l^*}{\partial \bar{\lambda}}.$$

Then, we can establish the following proposition regarding the effects of temporary increases in tax rates on the steady-state pollution stock.

Proposition 3 *(i) Assume that* $\Lambda > 0$. *Then, in the long run, a temporary increase in the consumption tax rate leads to smaller pollution stock, while a temporary increase in the income tax rate leads to larger pollution stock. (ii) Assume that* $\Lambda < 0$. *Then, in the long run, a temporary increase in the consumption tax rate leads to larger pollution stock, while a temporary increase in the income tax rate leads to smaller pollution stock.*

Proof. See Appendix C. ■

In contrast to the case with permanent tax changes, a temporary change in taxes does not generate a direct effect on the pollution stock after the repeal of its policy; hence, only the indirect effect, through a change in shadow value $\bar{\lambda} = L(\tau^c, \tau^y)$, remains in the long run.

Let us assume that the only sources of pollution are consumption activities ($\alpha_c > 0 = \alpha_y$) and that the consumption tax rate increases temporarily. This is a special case of Case (ii) in Proposition 3. The increase in τ^c decreases the shadow value of future consumption $\bar{\lambda}$. In summary, future consumption is substituted by current consumption. Since this effect continues until the new steady state even though the tax rate is returned to its original level, the value of current consumption relative to future consumption is high. Consequently, this increase in consumption leads to a higher level of pollution stock in the long run, despite the government having implemented a more stringent environmental policy by temporarily increasing the consumption tax.

Let us consider another polar case, in which the pollution flow is caused by production activities ($\alpha_c = 0 < \alpha_y$), and assume that the government increases the consumption tax rate temporarily. This is a special case of Case (i) in Proposition 3. Since the higher consumption tax leads to a lower shadow value of future consumption $\bar{\lambda}$, households do not want to work more because they have lower motivation to save. As a result, the economy's output decreases, and so does the pollution stock.

Finally, consider the general case in which pollution is generated by both consumption and production activities ($\alpha_c > 0$ and $\alpha_y > 0$). An increase in τ^c reduces $\bar{\lambda}$, which leads to an increase in consumption and a reduction in output. If the increase in the pollution flow from consumption is lower than the reduction in the pollution flow from output, the net effect is a reduction in the pollution stock in the long run. This case corresponds to Case (i) of Proposition 3, meaning that a more stringent environmental policy in the form of a higher consumption tax is effective for reducing pollution even if it is implemented temporarily. However, if the increase in the pollution flow from consumption is larger than the reduction in the pollution flow from output, which corresponds to Case (ii) of Proposition 3, the temporary increase in the consumption tax rate has an adverse effect: it increases the pollution stock in the long run.

The effects of a temporary increase in the rate of income tax can be interpreted similarly. As an important policy implication, a temporary increase in the income tax rate leads to larger pollution stock in the long run when the only sources of the pollution flow are production activities (i.e., $\alpha_c = 0 < \alpha_y$).

The abovementioned findings on the long-run effects of temporary policy highlight a possibility of government failure in controlling pollution. Needless to say, for given levels of pollution stock and the shadow value, an increase in the consumption tax unambiguously reduces consumption and thus, reduces consumption-generated pollution, and a similar reasoning can be applied to the case of an increase in the income tax. However, taking the dynamic adjustment of the shadow value into account, these tax policies can actually increase pollution if they are implemented temporarily. This is contrary to our expectations, and we should be more careful about the efficacy of pollution control policy since, as mentioned in the introduction of this paper, such policy schemes can be reverted due to political reasons.

Figure 3 illustrates the dynamics of the pollution stock and foreign assets after a temporary change in environmental policies, where E_0 shows the original steady state and P_0 is the corresponding pollution stock level. Figure 3(a) shows the case in which the sign of (#1) in (23) is negative, whereas Figure 3(b) illustrates the case in which (#1) is positive. In both figures, we focus on the "government failure" case, in which the temporary tax increases lead to larger pollution stock in the long run. For simplicity, we assume that the consumption tax rate increases temporarily and pollution is caused only by consumption (i.e., $\alpha_c > 0 = \alpha_y$). Then, the pollution stock decreases monotonically during the temporary policy implementation ($t \in [0, T)$) because of the downward shift in the Ω_t curve, as shown by (26a). The shift in the Ω_t curve, depicted by the dotted curve Ω_t' in the upper panel of Figure 3, reduces the pollution stock to the level of P_T at time T. However, after the repeal of the environmental policy at time T, the pollution stock does not decrease further, but starts to increase toward the new steady-state equilibrium, which differs from the original one. The reason is that when the consumption tax rate returns to its original level, the direct impact on the pollution stock disappears (i.e., $\partial c^*/\partial \tau^c = 0$ in (26a)), and thus, the Ω_t curve always shifts upward of its original position:

$$\left.\frac{\partial \Omega}{\partial \tau^c}\right|_{P_t=\text{const}} = \alpha_c G'(c^*)\frac{\partial c^*}{\partial \bar{\lambda}} L_{\tau^c} > 0, \tag{27}$$

where we assume that $\alpha_c > 0 = \alpha_y$. As shown in the upper panel of Figure 3, the Ω' curve shifts outward toward the Ω'' curve, which intersects with the θP_t line at the new steady state, E_1. The corresponding steady-state pollution stock, P_1^*, is larger than the original steady-state level P_0^*. The effect on the foreign assets depends on the sign of (#1) in (23). The lower panel of Figure 3(a) corresponds to the case in which (#1) is negative, and thus, the pollution stock and foreign assets have a negative relationship. In this case, the foreign assets initially increase, but after the repeal of the environmental policy, they decrease toward E_1. However, if the sign of (#1) in (23) is positive, the movement of the foreign assets is in the opposite direction.

[Figure 3 around here.]

5 Optimal tax policies

We have so far assumed that the government sets the consumption and income tax rates at some exogenous levels and considers the effects of exogenous changes in these tax rates. Since the small open economy faces aggregate pollution externalities and households do not consider their choice of consumption and labor supply on aggregate pollution in the decentralized market equilibrium, the decentralized equilibrium path with tax rates arbitrarily chosen fails to achieve a socially optimal resource allocation. Thus, we are interested in whether there is a time path of optimal taxes that achieves a socially optimal allocation and, if so, how such a path can be characterized.[10]

5.1 Socially optimal solution of the small open economy

Before analyzing optimal tax rules, we derive the socially optimal path determined by a social planner that considers the evolution of pollution (2), and thus, maximizes (4) subject to not only (8) but also (2). Let us denote the value of a variable x along the socially optimal path by "$\tilde{x}$" and define the current-value Hamiltonian associated with the social planner's dynamic optimization problem as follows:

$$\tilde{\mathcal{H}} = u(\tilde{c}_t, \tilde{P}_t) - \omega(\tilde{l}_t, \tilde{P}_t) + \tilde{\lambda}_t\left(r\tilde{b}_t + \tilde{l}_t f(\bar{k}) - \tilde{c}_t\right) - \tilde{\phi}_t\left(\alpha_c G(\tilde{c}_t) + \alpha_y N(\tilde{l}_t f(\bar{k})) - \theta \tilde{P}_t\right), \tag{28}$$

[10] Hereafter, we omit subscript j for ease of notation.

where $\tilde{\phi}_t$ indicates the shadow value of pollution associated with (2).

The first-order necessary conditions are as follows:

$$\tilde{H}_{\tilde{c}_t}: \ u_c(\tilde{c}_t, \tilde{P}_t) = \tilde{\lambda}_t + \tilde{\phi}_t \alpha_c G'(\tilde{c}_t), \tag{29a}$$

$$\tilde{H}_{\tilde{l}_t}: \ \omega_l(\tilde{l}_t, \tilde{P}_t) = \tilde{\lambda}_t f(\bar{k}) - \tilde{\phi}_t \alpha_y N'(\tilde{l}_t f(\bar{k})) f(\bar{k}), \tag{29b}$$

$$\tilde{H}_{\tilde{b}_t}: \ \tilde{\lambda}_t r = -\dot{\tilde{\lambda}}_t + \rho \tilde{\lambda}_t, \tag{29c}$$

$$\tilde{H}_{\tilde{P}_t}: \ u_P(\tilde{c}_t, \tilde{P}_t) - \omega_P(\tilde{l}_t, \tilde{P}_t) = \dot{\tilde{\phi}}_t - (\rho + \theta)\tilde{\phi}_t, \tag{29d}$$

In addition, the transversality conditions are given by

$$\lim_{t \to \infty} \tilde{\lambda}_t \tilde{b}_t e^{-\rho t} = 0, \quad \lim_{t \to \infty} \tilde{\phi}_t \tilde{P}_t e^{-\rho t} = 0. \tag{29e}$$

From (29a)–(29d), we can obtain the dynamic system of the economy consisting of state variables, namely, the aggregate pollution stock and foreign asset holdings and their respective shadow values. Specifically, as in the decentralized equilibrium, the assumption of a small open economy, $\rho = r$, yields the time-invariant shadow value $\tilde{\lambda} = \bar{\tilde{\lambda}}$ from (29c). Substituting $\tilde{\lambda}_t = \bar{\tilde{\lambda}}$ into (29a) and (29b), and solving for $\tilde{c}_t$ and $\tilde{l}_t$, respectively, we obtain the optimal levels of consumption and labor supply along the socially optimal path as follows:

$$\tilde{c}_t = \tilde{c}(\tilde{P}_t, \bar{\tilde{\lambda}}, \tilde{\phi}), \quad \tilde{l}_t = \tilde{l}(\tilde{P}_t, \bar{\tilde{\lambda}}, \tilde{\phi}_t), \tag{30}$$

where

$$\frac{\partial \tilde{c}_t}{\partial \tilde{P}_t} = -\frac{u_{cP}(\tilde{c}_t, \tilde{P}_t)}{\Theta_c}(< 0), \quad \frac{\partial \tilde{c}_t}{\partial \tilde{\phi}_t} = \frac{\alpha_c G'(\tilde{c}_t)}{\Theta_c}(< 0), \quad \frac{\partial \tilde{c}_t}{\partial \bar{\tilde{\lambda}}} = \frac{1}{\Theta_c}(< 0),$$
$$\frac{\partial \tilde{l}_t}{\partial \tilde{P}_t} = -\frac{\omega_{lP}(\tilde{l}_t, \tilde{P}_t)}{\Theta_l}(< 0), \quad \frac{\partial \tilde{l}_t}{\partial \tilde{\phi}_t} = -\frac{\alpha_y N'(\tilde{l}_t f(\bar{k})) f(\bar{k})}{\Theta_l}(< 0), \quad \frac{\partial \tilde{l}_t}{\partial \bar{\tilde{\lambda}}} = \frac{f(\bar{k})}{\Theta_l}(> 0),$$
$$\Theta_c \equiv u_{cc}(\tilde{c}_t, \tilde{P}_t) - \tilde{\phi}_t \alpha_c G''(\tilde{c}_t)(< 0), \quad \Theta_l \equiv \omega_{ll}(\tilde{l}_t, \tilde{P}_t) + \tilde{\phi}_t \alpha_y N''(\tilde{l}_t f(\bar{k})) f(\bar{k})^2 (> 0).$$

Substituting the optimal consumption and labor supply into (2) and (29d), it follows that

$$\dot{\tilde{P}}_t = \alpha_c G(\tilde{c}(\tilde{P}_t, \bar{\tilde{\lambda}}, \tilde{\phi}_t)) + \alpha_y N(\tilde{l}(\tilde{P}_t, \bar{\tilde{\lambda}}, \tilde{\phi}_t) f(\bar{k})) - \theta \tilde{P}_t, \tag{31a}$$

$$\dot{\tilde{\phi}}_t = (\rho + \theta)\tilde{\phi}_t + u_P(\tilde{c}(\tilde{P}_t, \bar{\tilde{\lambda}}, \tilde{\phi}_t), \tilde{P}_t) - \omega_P(\tilde{l}(\tilde{P}_t, \bar{\tilde{\lambda}}, \tilde{\phi}_t), \tilde{P}_t). \tag{31b}$$

In the steady state, $\dot{\tilde{P}}_t = \dot{\tilde{\phi}}_t = 0$ in (31) yields the following conditions:

$$\alpha_c G(\tilde{c}(\tilde{P}^*, \bar{\tilde{\lambda}}, \tilde{\phi}^*)) + \alpha_y N(\tilde{l}(\tilde{P}^*, \bar{\tilde{\lambda}}, \tilde{\phi}^*) f(\bar{k})) = \theta \tilde{P}^*. \tag{32a}$$

$$(\rho + \theta)\tilde{\phi}^* = \omega_P(\tilde{l}(\tilde{P}^*, \bar{\tilde{\lambda}}, \tilde{\phi}^*), \tilde{P}^*) - u_P(\tilde{c}(\tilde{P}^*, \bar{\tilde{\lambda}}, \tilde{\phi}^*), \tilde{P}^*). \tag{32b}$$

Note that the shadow value $\bar{\bar{\lambda}}$ should actually be solved endogenously from the equilibrium conditions. To obtain $\bar{\bar{\lambda}}$, let us focus on the stable root of the dynamic system in this economy. Using (32), we obtain the following:

$$\tilde{\mu} = \frac{1}{2}\Bigg\{\rho - \Bigg(\rho^2 - 4\underbrace{\bigg(\frac{\partial\dot{\tilde{\phi}}_t}{\partial\tilde{\phi}_t}\frac{\partial\dot{\tilde{P}}_t}{\partial\tilde{P}_t} - \frac{\partial\dot{\tilde{P}}_t}{\partial\tilde{\phi}_t}\frac{\partial\dot{\tilde{\phi}}_t}{\partial\tilde{P}_t}\bigg)}_{(\#2)}\Bigg)^{-\frac{1}{2}}\Bigg\}, \tag{33}$$

where

$$\frac{\partial\dot{\tilde{\phi}}_t}{\partial\tilde{\phi}_t} = \rho + \theta + u_{cP}(\tilde{c}^*, \tilde{P}^*)\frac{\partial\tilde{c}_t}{\partial\tilde{\phi}_t} - \omega_{lP}(\tilde{l}^*, \tilde{P}^*)\frac{\partial\tilde{l}_t}{\partial\tilde{\phi}_t}(> 0),$$

$$\frac{\partial\dot{\tilde{P}}_t}{\partial\tilde{P}_t} = \alpha_c G'(\tilde{c}^*)\frac{\partial\tilde{c}_t}{\partial\tilde{P}_t} + \alpha_y N'(\tilde{l}^* f(\bar{k}))f(\bar{k})\frac{\partial\tilde{l}_t}{\partial\tilde{P}_t} - \theta(< 0),$$

$$\frac{\partial\dot{\tilde{\phi}}_t}{\partial\tilde{P}_t} = u_{cP}(\tilde{c}^*, \tilde{P}^*)\frac{\partial\tilde{c}_t}{\partial\tilde{P}_t} + u_{PP}(\tilde{c}^*, \tilde{P}^*) - \omega_{lP}(\tilde{l}^*, \tilde{P}^*)\frac{\partial\tilde{l}_t}{\partial\tilde{P}_t} - \omega_{PP}(\tilde{l}^*, \tilde{P}^*),$$

$$\frac{\partial\dot{\tilde{P}}_t}{\partial\tilde{\phi}_t} = \alpha_c G'(\tilde{c}^*)\frac{\partial\tilde{c}_t}{\partial\tilde{\phi}_t} + \alpha_y N'(\tilde{l}^* f(\bar{k}))f(\bar{k})\frac{\partial\tilde{l}_t}{\partial\tilde{\phi}_t}(< 0).$$

We obtain the following proposition regarding the uniqueness of the steady state, and its stability can be summarized as follows.

Proposition 4 *(i) Assume that $\alpha_c > 0$ and $\alpha_y = 0$. Then, the steady-state equilibrium with saddle-path stability is uniquely determined if the following condition is satisfied:*

$$\Phi_l \equiv \omega_{PP}(\tilde{l}(\tilde{P}_t, \bar{\bar{\lambda}}, \tilde{\phi}_t), \tilde{P}^*) - u_{PP}(\tilde{c}(\tilde{P}^*, \bar{\bar{\lambda}}, \tilde{\phi}^*), \tilde{P}^*) + \omega_{lP}(\tilde{l}(\tilde{P}_t, \bar{\bar{\lambda}}, \tilde{\phi}_t), \tilde{P}^*)\frac{\partial\tilde{l}^*}{\partial\tilde{P}^*} \geq 0. \tag{34a}$$

(ii) Assume that $\alpha_c = 0$ and $\alpha_y > 0$. Then, the steady-state equilibrium with saddle-path stability is uniquely determined if the following condition is satisfied:

$$\Phi_c \equiv \omega_{PP}(\tilde{l}(\tilde{P}_t, \bar{\bar{\lambda}}, \tilde{\phi}_t), \tilde{P}^*) - u_{PP}(\tilde{c}(\tilde{P}^*, \bar{\bar{\lambda}}, \tilde{\phi}^*), \tilde{P}^*) - u_{cP}(\tilde{c}(\tilde{P}^*, \bar{\bar{\lambda}}, \tilde{\phi}^*), \tilde{P}^*)\frac{\partial\tilde{c}^*}{\partial\tilde{P}^*} \geq 0. \tag{34b}$$

Proof. See Appendix D. ■

The linearly approximated equations of aggregate pollution and its shadow value are given by:[11]

$$\tilde{P}_t = \tilde{P}^* + (P_0 - \tilde{P}^*)e^{\tilde{\mu}t}, \quad \tilde{\phi}_t = \tilde{\phi}^* + \tilde{A}(P_0 - \tilde{P}^*)e^{\tilde{\mu}t}, \tag{35}$$

[11]The initial values of state variables (i.e., pollution stock and foreign assets) are given constants, and we assume that these values are the same in the social planner's problem as those in the decentralized economy, that is, $\tilde{P}_0 = P_0$ and $\tilde{b}_0 = b_0$.

where $\tilde{A}$ is an element of eigenvector under the stable root (33) and is given by:[12]

$$\tilde{A} = \frac{\frac{\partial \dot{\tilde{\phi}}_t}{\partial \tilde{P}_t}}{\tilde{\mu} - \frac{\partial \dot{\tilde{\phi}}_t}{\partial \tilde{\phi}_t}}. \tag{36}$$

Substituting (30) into the dynamics of foreign assets, we have

$$\dot{\tilde{b}}_t = r\tilde{b}_t + l(\tilde{P}_t, \bar{\tilde{\lambda}}, \tilde{\phi}_t) f(\bar{k}) - c(\tilde{P}_t, \bar{\tilde{\lambda}}, \tilde{\phi}_t),$$

and linearly approximating this equation around the steady state and solving for it yields

$$\tilde{b}_t = \tilde{b}^* + \frac{f(\bar{k})\frac{\partial \tilde{l}_t}{\partial \tilde{P}_t} - \frac{\partial \tilde{c}_t}{\partial \tilde{P}_t} + \left(f(\bar{k})\frac{\partial \tilde{l}_t}{\partial \tilde{P}_t} - \frac{\partial \tilde{c}_t}{\partial \tilde{\phi}_t} \right) \tilde{A}}{\tilde{\mu} - r} (P_0 - \tilde{P}^*) e^{\tilde{\mu} t}. \tag{37}$$

Finally, the time-invariant solution of shadow value $\tilde{\lambda}$ is determined by the steady-state conditions $\dot{\tilde{P}}_t = \dot{\tilde{\phi}}_t = 0$ in (32) and the following two equations:

$$\dot{\tilde{b}}_t = 0: \quad r\tilde{b}^* + l(\tilde{P}^*, \bar{\tilde{\lambda}}, \tilde{\phi}^*) f(\bar{k}) = c(\tilde{P}^*, \bar{\tilde{\lambda}}, \tilde{\phi}^*), \tag{38a}$$

$$b_0 - \tilde{b}^* = \frac{f(\bar{k})\frac{\partial \tilde{l}_t}{\partial \tilde{P}_t} - \frac{\partial \tilde{c}_t}{\partial \tilde{P}_t} + \left(f(\bar{k})\frac{\partial \tilde{l}_t}{\partial \tilde{P}_t} - \frac{\partial \tilde{c}_t}{\partial \tilde{\phi}_t} \right) \tilde{A}}{\tilde{\mu} - r} (P_0 - \tilde{P}^*). \tag{38b}$$

Note that substituting $t = 0$ into (37), we obtain (38b).

5.2 Dynamic system of optimal tax policies

Here, we examine the optimal tax policies that achieve the socially optimal solution in the decentralized economy derived in the previous subsection. The equivalence between the decentralized equilibrium and socially optimal paths means that $c_t = \tilde{c}_t$, $b_t = \tilde{b}_t$, $l_t = \tilde{l}_t$, $P_t = \tilde{P}_t$, and $\bar{\lambda} = \bar{\tilde{\lambda}}$ (=constant) over time.

By comparing the first-order condition for optimal consumption in the decentralized equilibrium (11a) with that in the social optimum (29a) and using (13), we find that these equations coincide if the consumption tax rate is set as follows:

$$\tau_t^c = \frac{\alpha_c \tilde{\phi}_t G'(c(P_t, \bar{\lambda}, \tau_t^c))}{\bar{\lambda}}. \tag{39a}$$

Similarly, by comparing the first-order condition for optimal labor input in the decentralized equilibrium (11b) with that in the social optimum (29b) and using (15), these conditions coincide if the income tax rate is set as follows:

$$\tau_t^y = \frac{\alpha_y \tilde{\phi}_t N'(l(P_t, \bar{\lambda}, \tau_t^y) f(\bar{k}))}{\bar{\lambda}}. \tag{39b}$$

[12]Specifically, we calculate

$$\begin{bmatrix} \frac{\partial \dot{\tilde{\phi}}_t}{\partial \tilde{\phi}_t} - \tilde{\mu} & \frac{\partial \dot{\tilde{\phi}}_t}{\partial \tilde{P}_t} \\ \frac{\partial \dot{\tilde{P}}_t}{\partial \tilde{\phi}_t} & \frac{\partial \dot{\tilde{P}}_t}{\partial \tilde{P}_t} - \tilde{\mu} \end{bmatrix} \begin{bmatrix} \tilde{A} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, we obtain (36).

To obtain a clear exposition and intuition of the properties of optimal tax policies, we henceforth assume that the pollution flow is generated by either consumption ($\alpha_c > 0 = \alpha_y$) or production ($\alpha_c = 0 < \alpha_y$).

5.2.1 The case of $\alpha_c > 0$ and $\alpha_y = 0$

Suppose that pollution is generated only by consumption ($\alpha_c > 0$ and $\alpha_y = 0$). Then, from (39b), $\tau_t^y = 0$ holds over time. As a result, from (13) and (15), we have

$$c_t = c(P_t, \bar{\lambda}, \tau_t^c), \quad l_t = l(P_t, \bar{\lambda}, \tau_t^y) = l(P_t, \bar{\lambda}, 0) \equiv l(P_t, \bar{\lambda}). \tag{40}$$

Differentiating (39a) with respect to time and substituting (29d) into it, we obtain the dynamics of the consumption tax as follows:[13]

$$\dot{\tau}_t^c = \Delta_t^c \left(\rho + \theta + \alpha_c G'(c_t) \frac{u_P(c_t, P_t) - \omega_P(l_t, P_t)}{\tau_t^c \bar{\lambda}} + \frac{G''(c_t)}{G'(c_t)} \frac{\partial c_t}{\partial P_t} [\alpha_c G(c_t) - \theta P_t] \right), \tag{41a}$$

where c_t and l_t are given by (40) and

$$\Delta_t^c = \left(\frac{1}{\tau_t^c} - \frac{G''(c_t)}{G'(c_t)} \frac{\partial c_t}{\partial \tau_t^c} \right)^{-1} (> 0).$$

The dynamic equation of the pollution stock is given by

$$\dot{P}_t = \alpha_c G(c_t) - \theta P_t. \tag{41b}$$

The system of dynamic equations (41a) and (41b) characterize the path of τ_t^c and P_t in this economy, given the shadow value of foreign assets.

Given the shadow value $\bar{\lambda}$, the steady-state levels of pollution stock and consumption tax rate are determined by $\dot{\tau}_t^c = 0$ in (41a) and $\dot{P}_t = 0$ in (41b):

$$(\rho + \theta)\tau^{c,*}\bar{\lambda} = \alpha_c G'(c^*) \left\{ \omega_P(l^*, P^*) - u_P(c^*, P^*) \right\}, \tag{42a}$$

$$\alpha_c G(c^*) = \theta P^*, \tag{42b}$$

where c^* and l^* are determined by substituting P^* and $\tau^{c,*}$ into (40).

We obtain the following proposition regarding the properties of the steady state.[14]

Proposition 5 *Given the shadow value $\bar{\lambda}^*$, the steady-state levels of consumption tax and pollution stock are uniquely determined if condition* (34a) *is satisfied.*

[13] Differentiating (39a) with respect to time, we obtain:

$$\frac{\dot{\tilde{\phi}}_t}{\tilde{\phi}_t} = \left(\frac{1}{\tau_t^c} - \frac{G''(c_t)}{G'(c_t)} \frac{\partial c_t}{\partial \tau_t^c} \right) \dot{\tau}_t^c - \frac{G''(c_t)}{G'(c_t)} \frac{\partial c_t}{\partial P_t} \left[\alpha_c G(c_t) - \theta P_t \right].$$

[14] Once the steady-state levels of consumption tax and pollution stock are uniquely determined, the steady-state levels of foreign assets and its shadow value are determined by (38).

Proof. See Appendix E. ■

We characterize the socially optimal path by using a phase diagram in the (τ^c, P) space. For the $\dot{P}_t = 0$ locus, from (42b), we obtain

$$\left.\frac{\partial \tau^{c,*}}{\partial P^*}\right|_{\dot{P}_t=0} = \frac{\theta - \alpha_c G'(c^*)\frac{\partial c^*}{\partial P^*}}{\alpha_c G'(c^*)\frac{\partial c^*}{\partial \tau^{c,*}}} < 0, \tag{43}$$

where $\frac{\partial c^*}{\partial \tau^{c,*}}(< 0)$ and $\frac{\partial c^*}{\partial P^*}(< 0)$ are given by (13). From (42b), if the pollution stock goes to infinity, consumption goes to infinity as well. In this case, the marginal utility of consumption becomes 0 because of the Inada conditions in (5c). As a result, from (11a) and given a finite level of shadow value $\bar{\lambda}^*$, $\tau^{c,*} \to -1$ as $P^* \to \infty$. If the pollution stock goes to 0, from (42b), it follows that consumption becomes 0 as well. Hence, the marginal utility of consumption becomes infinity under (5c). Using (11a), we find that $\tau^{c,*} \to \infty$ as $P^* \to 0$. Therefore, the $\dot{P}_t = 0$ curve can be depicted as in Figure 4.

[Figure 4 around here.]

For the $\dot{\tau}^c = 0$ locus, totally differentiating (42a) and rearranging it yields

$$\left.\frac{\partial \tau^{c,*}}{\partial P^*}\right|_{\dot{\tau}^c_t=0} = \Omega^c \left(\underbrace{\frac{G''(c^*)}{G'(c^*)}\frac{\partial c^*}{\partial P^*}}_{(-)} + \frac{\Phi_l - u_{cP}(c^*, P^*)\frac{\partial \tilde{c}^*}{\partial \tilde{P}^*}}{\omega_P(l^*, P^*) - u_P(c^*, P^*)} \right), \tag{44}$$

where Φ_l is defined by (34a) and

$$\Omega^c = \left(\frac{1}{\tau^{c,*}} - \left(\frac{G''(c^*)}{G'(c^*)} - \frac{u_{cP}(c^*, P^*)}{\omega_P(l^*, P^*) - u_P(c^*, P^*)} \right) \frac{\partial c^*}{\partial \tau^{c,*}} \right)^{-1} > 0.$$

Although the sign of the terms between the parentheses in (44) is unclear, we note that the $\dot{\tau}^c_t$ locus is located in the region in which $\tau_c > 0$ holds, as per Figure 4. This is because the right-hand side of (42a) is always positive. We also find the following.

Remark 1 *Under condition* (34a)*, it holds that* $\left.\frac{\partial \tau^{c,*}}{\partial P^*}\right|_{\dot{P}_t=0} < \left.\frac{\partial \tau^{c,*}}{\partial P^*}\right|_{\dot{\tau}^c_t=0}$.

Proof. Comparing each slope in (43) and (44), it follows that

$$\left.\frac{d\tau^{c,*}}{dP^*}\right|_{\dot{P}_t=0} - \left.\frac{d\tau^{c,*}}{dP^*}\right|_{\dot{\tau}^c_t=0} = \underbrace{\frac{\Omega^c}{\alpha_c G' \frac{\partial c^*}{\partial \tau^{c,*}}}}_{(-)} \times \left\{ \underbrace{\theta(\Omega^c)^{-1}}_{(+)} + \underbrace{\frac{\alpha_c G' u_{cP}}{u_c}\frac{\partial c^*}{\partial \tau^{c,*}}}_{(+)} \underbrace{\left(\frac{1+\tau^{c,*}}{\tau^{c,*}} - \frac{u_c \Phi_l}{u_{cP}(\omega_P - u_P)} \right)}_{(+)} \right\} < 0.$$

This completes the proof. ■

Based on Proposition 5 and Remark 1, we depict the $\dot{\tau}^c_t = 0$ locus as shown in Figure 4(a). Let us assume for the moment that the sum of the terms between the parentheses in (44) has a negative sign. In this case, the slope of the $\dot{\tau}^c_t = 0$ locus is negative and not steeper than that of the $\dot{P}_t = 0$ locus, as in Figure 4(a). At the same time, we can obtain $\frac{\partial \dot{\tau}_t^c}{\partial P_t} > 0$ and $\frac{\partial \dot{P}_t}{\partial \tau_t^c} < 0$.[15] As a result, the phase diagram is as in Figure 4(a).

To understand the logic of Figure 4(a), we identify the condition under which the sum of the terms between the parentheses in (44) is negative. Since the first term between the parentheses is negative, (44) is likely to be negative if $u_{cp}\frac{\partial c^*}{\partial P^*}$ is sufficiently large. In other words, when the distaste effect has a significant impact, we may have a phase diagram as in Figure 4(a): if the economy starts from a low level of pollution stock, P_0, the consumption tax rate decreases over time as the pollution stock increases along the optimal path. This may be contrary to our standard intuition that a higher level of pollution should be accompanied by higher pollution taxes to reduce pollution. This is due to the distaste effect being sufficiently large. In this case, an increase in pollution leads to a large decrease in the marginal utility of consumption, which then implies an increase in consumption. To achieve a higher consumption level, it is optimal to reduce the consumption tax rate. Thus, τ^c is decreasing in P along the optimal path.

If we assume, instead, that the distaste effect is not very large, the terms between the parentheses in (44) are positive. Therefore, as shown in Figure 4(b), the optimal consumption tax increases as the pollution stock increases. The mechanism is opposite to that in Figure 4(a).

5.2.2 The case of $\alpha_c = 0$ and $\alpha_y > 0$

We now consider the case in which pollution is generated only by production activities ($\alpha_c = 0$ and $\alpha_y > 0$). From (39a), it follows that $\tau_t^c = 0$, and thus,

$$c_t = c(P_t, \bar{\lambda}, \tau_t^c) = c(P_t, \bar{\lambda}, 0) \equiv c(P_t, \bar{\lambda}), \quad l_t = l(P_t, \bar{\lambda}, \tau_t^y). \tag{45}$$

[15] We can show that:

$$\frac{\partial \dot{\tau}_t^c}{\partial \tau_t^c} = \Delta^{c,*}\left((\rho+\theta)\left(\frac{1}{\tau^{c,*}} - \left(\frac{G''}{G'} - \frac{u_{cP}}{\omega_P - u_P}\right)\frac{\partial c^*}{\partial \tau^{c,*}}\right) + \alpha_c G'' \frac{\partial c^*}{\partial P^*}\frac{\partial c^*}{\partial \tau^{c,*}}\right) > 0,$$

$$\frac{\partial \dot{\tau}_t^c}{\partial P_t} = \Delta^{c,*}\left(-(\rho+\theta)\left(\frac{G''}{G'}\frac{\partial c^*}{\partial P^*} + \frac{\Phi_l - u_{cP}\frac{\partial c^*}{\partial P^*}}{\omega_P - u_P}\right) + \frac{G''}{G'}\frac{\partial c^*}{\partial P^*}\left(\alpha_c G' \frac{\partial c^*}{\partial P^*} - \theta\right)\right) > 0,$$

$$\frac{\partial \dot{P}_t}{\partial \tau_t^c} = \alpha_c G' \frac{\partial c^*}{\partial \tau^{c,*}} < 0, \quad \frac{\partial \dot{P}_t}{\partial P_t} = \alpha_c G' \frac{\partial c^*}{\partial P^*} - \theta < 0.$$

Differentiating (39b) with respect to time and substituting (29d) into it, we obtain the following dynamic equation of the income tax rate:[16]

$$\dot{\tau}_t^y = \Delta_t^y \left(\rho + \theta + \alpha_y N'(l_t f(\bar{k})) \frac{u_p(c_t, P_t) - \omega_P(l_t, P_t)}{\tau_t^y \bar{\lambda}} + \frac{N''(l_t f(\bar{k})) f(\bar{k})}{N'(l_t f(\bar{k}))} \frac{\partial l_t}{\partial P_t} [\alpha_y N(l_t f(\bar{k})) - \theta P_t] \right), \tag{46}$$

where c_t and l_t are given by (45) and

$$\Delta_t^y = \left(\frac{1}{\tau_t^y} - \frac{N''(l_t f(\bar{k})) f(\bar{k})}{N'(l_t f(\bar{k}))} \frac{\partial l_t}{\partial \tau_t^y} \right)^{-1} (> 0).$$

The dynamic equation of the pollution stock is

$$\dot{P}_t = \alpha_y N \left(l_t f(\bar{k}) \right) - \theta P_t. \tag{47}$$

The optimal solution is characterized by the system of dynamic equations (46) and (47). Therefore, in the steady state, the following system of equations holds:

$$(\rho + \theta) \tau^{y,*} \bar{\lambda}^* = \alpha_y N' \left(l^* f(\bar{k}) \right) \{ \omega_P(l^*, P^*) - u_P(c^*, P^*) \}, \tag{48a}$$

$$\alpha_y N \left(l^* f(\bar{k}) \right) = \theta P^*. \tag{48b}$$

Proposition 6 *Given the shadow value* $\bar{\lambda}^*$*, the steady-state levels of income tax and pollution stock are uniquely determined if condition* (34b) *is satisfied.*

Proof. See Appendix E. ■

We describe the optimal path using a phase diagram. From (48b), the slope of the $\dot{P}_t = 0$ locus is derived as follows:

$$\left. \frac{\partial \tau^{y,*}}{\partial P^*} \right|_{\dot{P}_t = 0} = \frac{\theta - \alpha_y N'(l^* f(\bar{k})) f(\bar{k}) \frac{\partial l^*}{\partial P^*}}{\alpha_y N'(l^* f(\bar{k})) f(\bar{k}) \frac{\partial l^*}{\partial \tau^{y,*}}} < 0, \tag{49}$$

where $\frac{\partial l^*}{\partial \tau^{y,*}} < 0$ and $\frac{\partial l^*}{\partial P^*} < 0$ from (15). Suppose that the pollution stock goes to 0. Then, (48b) indicates that $N \left(l^* f(\bar{k}) \right)$ approaches 0, and so does the labor supply. In this case, (5c) indicates that the marginal disutility of the labor supply is 0, and thus, in light of (11b), $\tau^{y,*} \to 1$ as $P^* \to 0$. By contrast, if the pollution stock approaches infinity, the labor supply must go to infinity as well. In this case, because the marginal disutility of the labor supply is infinite in (5c), we can conclude that $\tau^{y,*} \to -\infty$ as $P^* \to \infty$. As a result, we obtain the shape of the $\dot{P}_t = 0$ locus, as shown in Figure 5.

[16]We differentiate (39b) with respect to time as follows:

$$\frac{\dot{\tilde{\phi}}_t}{\tilde{\phi}_t} = \left(\frac{1}{\tau_t^y} - \frac{N''(l_t f(\bar{k})) f(\bar{k})}{N'(l_t f(\bar{k}))} \frac{\partial l_t}{\partial \tau_t^y} \right) \dot{\tau}_t^y - \frac{N''(l_t f(\bar{k})) f(\bar{k})}{N'(l_t f(\bar{k}))} \frac{\partial l_t}{\partial P_t} \left[\alpha_y N(l_t f(\bar{k})) - \theta P_t \right].$$

[Figure 5 around here.]

We next consider the slope of the $\dot{\tau}_t^y = 0$ locus:

$$\frac{\partial \tau^{y,*}}{\partial P^*}\Big|_{\dot{\tau}_t^y=0} = \Omega^y \Bigg(\underbrace{\frac{N''(l^* f(\bar{k}))f(\bar{k})}{N'(l^* f(\bar{k}))} \frac{\partial l^*}{\partial P^*}}_{(-)} + \frac{\Phi_c + \omega_{lP}(l^*, P^*)\frac{\partial l^*}{\partial P^*}}{\omega_P(l^*, P^*) - u_P(c^*, P^*)} \Bigg), \tag{50}$$

where Φ_c is defined by (34b) and

$$\Omega^y = \left(\frac{1}{\tau^{y,*}} - \left(\frac{N''(l^* f(\bar{k})f(\bar{k})}{N'(l^* f(\bar{k}))} + \frac{\omega_{lP}(l^*, P^*)}{\omega_P(l^*, P^*) - u_P(c^*, P^*)} \right) \frac{\partial l^*}{\partial \tau^{y,*}} \right)^{-1} > 0.$$

The sign of (50) may be positive or negative. However, since the right-hand side of (48a) is always positive, the left-hand side must be positive as well, which means that the income tax rate falls between 0 and 1.

Similar to Remark 1, we obtain the following.

Remark 2 *Under condition* (34b)*, it holds that* $\frac{\partial \tau^{y,*}}{\partial P^*}\big|_{\dot{P}_t=0} < \frac{\partial \tau^{y,*}}{\partial P^*}\big|_{\dot{\tau}_t^y=0}$.

Proof. By using (34b), we have

$$\frac{\partial \tau^{y,*}}{\partial P^*}\Big|_{\dot{P}_t=0} - \frac{\partial \tau^{y,*}}{\partial P^*}\Big|_{\dot{\tau}_t^y=0} = \underbrace{\frac{\Omega^y}{\alpha N' f \frac{\partial l^*}{\partial \tau^y}}}_{(-)} \Bigg\{ \underbrace{\theta(\Omega^y)^{-1}}_{(+)} - \underbrace{\frac{\alpha_y N' f \omega_{lP}}{\omega_l} \frac{\partial l^*}{\partial \tau^{y,*}}}_{(+)} \underbrace{\left(\frac{1-\tau^{y,*}}{\tau^{y,*}} + \frac{\omega_l \Phi_c}{\omega_{lP}(\omega_P - u_P)} \right)}_{(+)} \Bigg\} < 0. \tag{51}$$

This completes the proof. ■

The properties of the $\dot{\tau}_t^y = 0$ locus in Figure 5(a) can now be described as follows. Assume that the sum of the terms between the parentheses in (50) is negative, so that the $\dot{\tau}_t^y = 0$ locus has a negative slope. In addition, Proposition 6 and Remark 2 indicate that the relationship between the $\dot{P}_t = 0$ and $\dot{\tau}_t^y = 0$ loci can be represented as in Figure 5(a). Moreover, $\frac{\partial \dot{\tau}_t^y}{\partial P_t} > 0$ and $\frac{\partial \dot{P}_t}{\partial \tau_t^y} < 0$ hold in this case. Therefore, we obtain the phase diagram in Figure 5(a).[17]

The intuition behind the dynamics of the optimal income tax is similar to that of the optimal consumption tax. When the values of the first term and $\omega_{lP} \frac{\partial l^*}{\partial P^*}$ between the parentheses in (50) are large enough for the sign of this equation to be negative, the optimal income tax decreases as the pollution stock increases. Intuitively, if the leisure effect is sufficiently large, the marginal

[17] Each derivative is

$$\frac{\partial \dot{\tau}_t^y}{\partial \tau_t^y} = \Delta^{y,*} \Bigg\{ (\rho+\theta)\left(\frac{1}{\tau^{y,*}} - \left(\frac{N'' f}{N'} + \frac{\omega_{lP}}{\omega_P - u_P} \right) \frac{\partial l^*}{\partial \tau^{y,*}} \right) + \alpha_y N'' f^2 \frac{\partial l^*}{\partial P^*} \frac{\partial l^*}{\partial \tau^{y,*}} \Bigg\} > 0,$$

$$\frac{\partial \dot{\tau}_t^y}{\partial P_t} = \Delta^{y,*} \Bigg(-(\rho+\theta)\left(\frac{N'' f}{N'} \frac{\partial l^*}{\partial P^*} + \frac{\Phi_c + \omega_{lP} \frac{\partial l^*}{\partial P^*}}{\omega_P - u_P} \right) + \frac{N'' f}{N'} \frac{\partial l^*}{\partial P^*} \left(\alpha_y N' f \frac{\partial l^*}{\partial P^*} - \theta \right) \Bigg) > 0,$$

$$\frac{\partial \dot{P}_t}{\partial \tau_t^y} = \alpha_y N' f \frac{\partial l^*}{\partial \tau^{y,*}} (< 0), \quad \frac{\partial \dot{P}_t}{\partial P_t} = \alpha_y N' f \frac{\partial l^*}{\partial P^*} - \theta < 0.$$

disutility of labor input increases to a large degree, and thus, there is a considerable decrease in income. This means that consumption will decline. As a result, the optimal rate of income tax decreases to avoid a decrease in consumption.

If the leisure effect is not strong, the sign of (50) may be positive, which corresponds to Figure 5(b). Then, the larger the pollution stock is, the larger is the optimal rate of income tax.

6 Numerical analysis

This section conducts numerical analysis to obtain further insights into the effects of environmental tax policies. Propositions 2 and 3 indicate that a temporary increase in consumption or income taxes may increase the pollution stock in the long run, while a permanent increase in these taxes unambiguously reduces the pollution stock in the long run. Consequently, we are interested in examining the quantitative impacts of these tax policies on the pollution stock, as well as the entire economy. Under the same parameter values, we are also interested in the rates of the optimal consumption and income taxes.

In the following, we focus on the case in which pollution is generated by either consumption or production activities; specifically, we assume either the case in which $\alpha_c = 1$ and $\alpha_y = 0$ or in which $\alpha_c = 0$ and $\alpha_y = 1$. In addition, we use the specific form of utility functions given by (7). Moreover, the pollution functions $G(c_t)$ and $N(l_t f(\bar{k}))$ are specified by $G(c_t) = c_t^{1+\chi}/(1+\chi)$ and $N(l_t f(\bar{k})) = (l_t f(\bar{k}))^{1+\zeta}/(1+\zeta)$, respectively, where $\chi > 0$ and $\zeta > 0$. The production function is given in a Cobb–Douglas form: $f(\bar{k}) = A\bar{k}^d$ where $A > 0$ and $d > 0$. Thus, the capital stock level is derived as $\bar{k} = (r/d)^{1/(d-1)}$.

The baseline parameters we use are summarized as follows:

Production parameters: $A = 1$, $d = 0.35$.

Preference parameters: $\rho = 0.1$, $\gamma = 0.8$, $\bar{h} = 50$, $\eta = 0.25$, $\delta = 2.5$, $\beta = 1.25$, $\epsilon = 0.25$.

Pollution parameters: $\chi = \zeta = 0.25$, $\theta = 0.05$.

Tax rates: $\tau_0^c = \tau_0^y = \tau_2^c = \tau_2^y = 0$, $\tau_1^c = \tau_1^y = 0.05$.

Enforcement duration of temporary policies: $T = 2$,

Initial value of foreign assets: $b_0 = b_0^* = 1$.

The choice of the production elasticity of labor measured in efficiency units, $\alpha = 0.35$, implies that 65% of output accrues to labor. The rate of time preference is set at 10% and, from our assumption of a small open economy, the return to capital, r, is also 10%. Therefore, we have

$\bar{k} \fallingdotseq 6.87$. To meet $\partial u(c_t, P_t)/\partial P_t < 0$ and $\partial^2 u(c_t, P_t)/\partial P_t^2 < 0$, the risk parameter must satisfy $0 < \gamma < 1$; we assume $\gamma = 0.8$. Parameters η and $\bar{h}$, which determine the scale of the distaste effect, are set to meet $\bar{h}^{1/(1+\eta)}(= 22.865) > P_t$ over time so that function $u(\cdot)$ satisfies (5a) and (6). Specifically, we set $\eta = 0.25$ and $\bar{h} = 50$. Parameters δ, β, and ϵ in function $\omega(\cdot)$ are chosen to satisfy $\delta(1+\epsilon) > 1$ and $\beta(1+\epsilon) > 1$, so that assumptions (5b) and (6) are met. Parameter θ is chosen by 0.05; that is, 5% of the pollution stock is purified at each moment. For simplicity, we assume that the elasticity parameters in the pollution functions are the same, irrespective of the pollution causes: $\chi = \zeta = 0.25$. Finally, the initial tax rates are set at $\tau_0^c = \tau_0^y = 0$. The government sets tax rates of 5% as the baseline tax policy, while the duration of the temporary increase in the respective rates is given by $T = 2$.

For the analysis of environmental policies in a decentralized economy, we assume that the economy is initially at the steady state and the initial level of foreign assets is $b_0 = b_0^* = 1$. The initial steady-state values of pollution stock P_0^* and shadow value of foreign assets $\bar{\lambda}_0$ are determined from equations (18) and (24). On the other hand, for the analysis of the optimal rates of consumption and income taxes, we choose the initial values of state variables P_0 and b_0 exogenously.

In addition to the baseline parameter set, we consider the following alternative sets of parameters:

Case (i): $\rho = 0.1 \Rightarrow \rho = 0.2$.

Case (ii): $\theta = 0.05 \Rightarrow \theta = 0.1$.

Case (iii): $A = 1 \Rightarrow A = 1.5$.

In Case (i), we assume that the rate of time preference increases from 10% to 20%, which means that households become more impatient. This makes households prefer current to future consumption, increasing the shadow value of future consumption. It also reduces savings, and thus, leads to a lower level of consumption in the steady state. In Case (ii), we consider the situation in which the decay rate of pollution increases from 5% to 10%, and more pollution can be cleaned up naturally. This means that the economy allows more pollution flow, and thus, economic agents increase consumption and labor supply. An increase in θ also leads the pollution stock to be lower than that under the baseline parameters. Case (iii) assumes an increase in the total factor productivity of 50%. Since larger output can be produced with less labor, the steady-state pollution stock and consumption will increase.

Table I gives the steady-state levels of $\{b_0^*, P_0^*, \bar{\lambda}_0^*, c_0^*, l_0^*\}$ before the policy changes at time 0 (i.e., $\tau_0^c = \tau_0^y = 0$) where it is assumed in Table I(a) that $\alpha_c > 0$ and $\alpha_y = 0$, and in Table I(b) that $\alpha_c = 0$ and $\alpha_y > 0$.

[Table I around here.]

6.1 The case in which $\alpha_c > 0$ and $\alpha_y = 0$

We begin with the case in which the pollution flow is generated only by consumption activities. For simplicity, we assume that the income tax rate is 0 over time (i.e., $\tau_t^y = 0$). Table II(a) shows the comparative static results of a permanent increase in the consumption tax rate from $\tau_0^c = 0$ to $\tau_1^c = 0.05$. For baseline parameters, we show that the long-run pollution stock decreases by around 0.6% (from $P_0^* = 15.934$ to $P_1^* = 15.8683$) as a result of a 0.50% decrease in consumption. Since not only consumption but also labor supply decreases, the steady-state level of foreign assets may increase or decrease, as given in (22). From Table II(a), we confirm an increase in foreign assets of around 4% ($b_0^* = 1$ and $b_1^* = 1.0398$). Finally, because an increase in the consumption tax rate increases the value of the current consumption relative to that of future consumption, the shadow value of future consumption $\bar{\lambda}$ decreases by around 3%.

[Table II around here.]

From Tables I(a) and II(a), Cases (i)–(iii) show that the pollution stock at the new steady state is lower than its original level, which is consistent with Proposition 2. Furthermore, increasing the consumption tax rate decreases the long-run levels of consumption, labor supply, and shadow value, while it increases foreign assets.

Table III(a) shows the simulation results for which the consumption tax rate temporarily increases from 0 to 5%. We find that $P_T < P_0^* < P_2^*$. Along the stable root of the dynamic equilibrium path, the pollution stock monotonically decreases during policy enforcement and, after the tax rate returns to its original level, the pollution stock begins to increase and finally surpasses its original level, as shown in Figures 3(a) and 3(b). Under the baseline parameters, foreign assets increase until the repeal of the tax policy and thereafter decrease until the new steady state, which is consistent with the lower panel of Figure 3(a).

[Table III around here.]

The temporary tax policy has no quantitatively significant impacts on the long-run levels of economic variables. Under the baseline parameters set, Table III(a) shows that, in response to the temporary increase in the consumption tax rate, the long-run pollution stock increases by around 0.025% (i.e., from $P_0^* = 15.934$ to $P_2^* = 15.938$). Since the levels of consumption and labor supply increase by 0.02% and 0.05%, respectively, the long-run level of foreign assets is reduced by 0.2% (from $b_0^* = 1$ to $b_2^* = 0.99799$). However, at time T, the levels of the pollution stock and foreign assets differ from their original levels to some extent. For example, when the consumption tax rate returns to its original level, the pollution stock at time T decreases by 0.6%.

Figure 6 helps better understand the effects of the temporary change in the tax policy. Specifically, Figure 6(a) shows the relationship between the consumption tax rate and pollution stock, where τ^c is between 0.01 and 0.99. The solid curve with "+" markers corresponds to P_1^* for the permanent increase in τ^c, and the dashed and dotted curves show P_2^* and P_T, respectively. Figure 6(a) shows that, even if the consumption tax rate is high, its temporary increase has a negligible impact on the long-run pollution stock. By contrast, P_1^* and P_T decrease as τ^c increases.

[Figure 6 around here.]

Figure 6(b) shows P_2^* (the lines with the circle and plus markers) and P_T (the dotted curves) when we consider longer enforcement duration of the tax policy T. Even if the enforcement duration increases, the increase in τ^c does not have significant impacts on the long-run pollution stock. This finding is robust to tax rates; observe the lines with the circle and the plus markers, which are almost the same as the original level. However, looking at the red/blue-colored broken curve, the pollution stock at time T decreases monotonically as the enforcement duration of the tax policy is longer or the consumption tax rate increases.

Finally, Table IV(a) presents the optimal rate of consumption tax.[18] The initial values of the state variables, b_0 and P_0, are set as in Table I(a). We find that the optimal rates of consumption tax are over 100%. For instance, under the baseline parameters, $\tau_0^c = 30.345$ and $\tau^{c,*} = 31.735$, implying that the optimal rate of consumption tax is over 3,000% at the initial period and thereafter increases toward the steady-state equilibrium. Then, the pollution stock decreases by 49.6%. With the adjustment of the optimal consumption tax, the pollution stock decreases significantly, which suggests that the decentralized equilibrium is significantly distorted by pollution externalities.

[Table IV around here.]

[18]We confirm that, under our parameter specifications, the optimal path is saddle-point stable.

6.2 The case in which $\alpha_c = 0$ and $\alpha_y > 0$

Let us assume that $\alpha_c = 0$ and $\alpha_y = 1$ and that the government implements only the income tax policy (i.e., $\tau_t^y > 0 = \tau_t^c$). Table II(b) shows the impacts on the main variables by increasing the income tax rate permanently. Under the baseline parameters, the long-run pollution stock decreases by around 1% (from $P_0^* = 13.025$ to $P_1^* = 12.894$) as a result of around a 0.8% decrease in the labor supply. Since the decrease in the labor supply reduces labor income, the shadow value of the future consumption increases (from $\bar{\lambda}_0^* = 26.351$ to $\bar{\lambda}_1^* = 26.755$) and consumption decreases (from $c_0^* = 0.94827$ to $c_1^* = 0.94475$). In Cases (i)–(iii), the long-run pollution stock is lower than the original steady-state level.

Regarding the temporary increase in the income tax rate, Table IV(b) shows that the observations in the previous subsection are valid. In summary, a temporary increase in the income tax rate does not have significant impacts on the long-run pollution stock. For example, under the baseline parameters, the long-run pollution stock increases slightly (from $P_0^* = 13.025$ to $P_2^* = 13.03$ in Table III(b)) as a result of the slight increase in the labor supply (from $l_0^* = 0.429$ to $l_2^* = 0.4322$). Figure 7 illustrates that this finding is robust, even if the income tax rate increases or the enforcement period of the tax policy is longer.

[Figure 7 around here.]

Finally, Table IV(b) shows the optimal rate of income tax. As in the previous subsection, the impacts of pollution externalities on the economy seem large because, for the adjustment of the optimal income tax, the pollution stock decreases to a large degree. Under the baseline parameters, the optimal rate of income tax is around 92.5% in the initial period and 93.6% in the long run, while the pollution stock decreases by around 42%.

7 Conclusions

This study examined the effects of tax policies in a dynamic model of a polluted small open economy with perfect international capital market access. In this economy, there are two sources of pollution flow—consumption and production—controlled by consumption and income taxes, while accumulated pollution has a negative effect on households' utility.

We began by analyzing a decentralized dynamic competitive equilibrium under exogenous tax rates and showed that a permanent increase in both the consumption and income taxes unambiguously reduces pollution in the long run. However, if the government implements tax policies only temporarily, there are cases in which an increase in these taxes may increase the pollution stock in the long run. Such adverse effects of environmental policies imply the possibility of *government failure*, which has a significant implication for the design of environmental policies.

We considered not only the decentralized equilibrium but also the socially optimal solution, and analyzed in detail the paths of consumption and income taxes that achieve the social optimum. We showed that whether pollution and tax rates are negatively or positively correlated along the optimal path depends on the strength of the distaste and leisure effects. If these effects are sufficiently strong, tax rates decrease along the optimal path as pollution increases over time, while if these effects are not so strong, the opposite changes occur. Our result suggests that, when designing a flexible environmental policy, if possible, the government must be conscious of households' preferences regarding the relationship between pollution and consumption or leisure.

In the analysis of optimal tax paths, we assumed that the source of pollution is either consumption or production and that the government uses one policy instrument. Therefore, we could improve our analysis by considering the case in which there are two sources of pollution and the government uses both consumption and income taxes. In such a case, there are inevitably two control variables, and thus, phase diagrams cannot be used. To determine the properties of optimal taxes, we have to solve the model numerically, which is left for future studies.

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Table I: Initial values in comparative static analysis $\tau_0^c = \tau_0^y = 0$

	(a) $\alpha_c = 1$ and $\alpha_y = 0$					(b) $\alpha_c = 0$ and $\alpha_y = 1$				
	b_0^*	P_0^*	$\bar{\lambda}_0^*$	c_0^*	l_0^*	b_0^*	P_0^*	$\bar{\lambda}_0^*$	c_0^*	l_0^*
Baseline	1	15.934	18.214	0.99668	0.45675	1	13.025	26.351	0.94827	0.43209
Case (i)	1	13.718	26.043	0.88416	0.50616	1	10.16	34.805	0.89538	0.51447
Case (ii)	1	11.623	22.471	1.3483	0.63584	1	9.5154	27.866	1.2489	0.58521
Case (iii)	1	20.294	5.949	1.2095	0.30286	1	17.849	11.571	1.1914	0.29794

Table II: Permanent increase of taxes by 5%

	(a) $\tau_1^c = 0.05$ and $\tau_1^y = 0$ ($\alpha_c = 1$ and $\alpha_y = 0$)					(b) $\tau_1^y = 0.05$ and $\tau_1^c = 0$ ($\alpha_c = 0$ and $\alpha_y = 1$)				
	b_1^*	P_1^*	$\bar{\lambda}_1^*$	c_1^*	l_1^*	b_1^*	P_1^*	$\bar{\lambda}_1^*$	c_1^*	l_1^*
Baseline	1.0398	15.834	17.655	0.99168	0.45218	1.0331	12.894	26.755	0.94475	0.42861
Case (i)	1.0183	13.63	25.129	0.87961	0.50008	11.0061	10.049	35.228	0.89051	0.50996
Case (ii)	1.017	11.516	21.714	1.3383	0.62991	1.0001	9.415	28.23	1.2392	0.58026
Case (iii)	1.0483	20.225	5.8269	1.2062	0.30066	1.1622	17.703	11.827	1.2005	0.29598

Table III(a): Temporary increase in τ^c

	b_T	P_T	b_2^*	P_2^*	$\bar{\lambda}_2^*$	c_2^*	l_2^*
Baseline	1.014	15.898	0.99799	15.938	18.205	0.99691	0.45697
Case (i)	1.005	13.694	0.99887	13.723	26.03	0.88444	0.50654
Case (ii)	1.0066	11.581	0.99924	11.626	22.465	1.3486	0.63604
Case (iii)	1.0348	20.245	0.998	20.296	5.9448	1.2096	0.30295

Table III(b): Temporary increase in τ^y

	b_T	P_T	b_2^*	P_2^*	$\bar{\lambda}_2^*$	c_2^*	l_2^*
Baseline	1.0063	13	0.9985	13.03	26.36	0.9484	0.4322
Case (i)	1.0012	10.139	0.99941	10.167	34.822	0.8956	0.5147
Case (ii)	1.0013	9.481	1.0016	9.5192	27.875	1.2494	0.5854
Case (iii)	1.0308	17.821	0.99655	17.851	11.573	1.1912	0.298

Table IV(a): Rate of the optimal consumption tax

	b_0	P_0	τ_0^c	$\tau^{c,*}$	b^*	P^*	$\bar{\lambda}^*$	c^*	l^*
Baseline	1	15.934	30.345	31.735	0.84993	8.0258	1.7336	0.57583	0.25002
Case (i)	1	13.718	6.5978	6.9104	1.3618	10.01	5.4951	0.68712	0.30686
Case (ii)	1	11.623	14.492	14.633	0.00051	6.1625	3.0976	0.79468	0.39899
Case (iii)	1	20.294	196.6	210.7	0.85324	8.1725	0.26275	0.58419	0.13617

Table IV(b): Rate of the optimal income tax

	b_0	P_0	τ_0^y	$\tau^{y,*}$	b^*	P^*	$\bar{\lambda}^*$	c^*	l^*
Baseline	1	13.025	0.92501	0.93571	0.33544	7.5349	57.925	0.58102	0.27887
Case (i)	1	10.16	0.72172	0.73999	0.88855	7.6258	47.99	0.73046	0.40894
Case (ii)	1	9.5154	0.89722	0.90511	0.12334	5.7924	49.794	0.78468	0.39342
Case (iii)	1	17.849	0.98342	0.98755	0.00245	8.1711	55.987	0.57949	0.15829

Figure 1: Decentralized equilibrium path

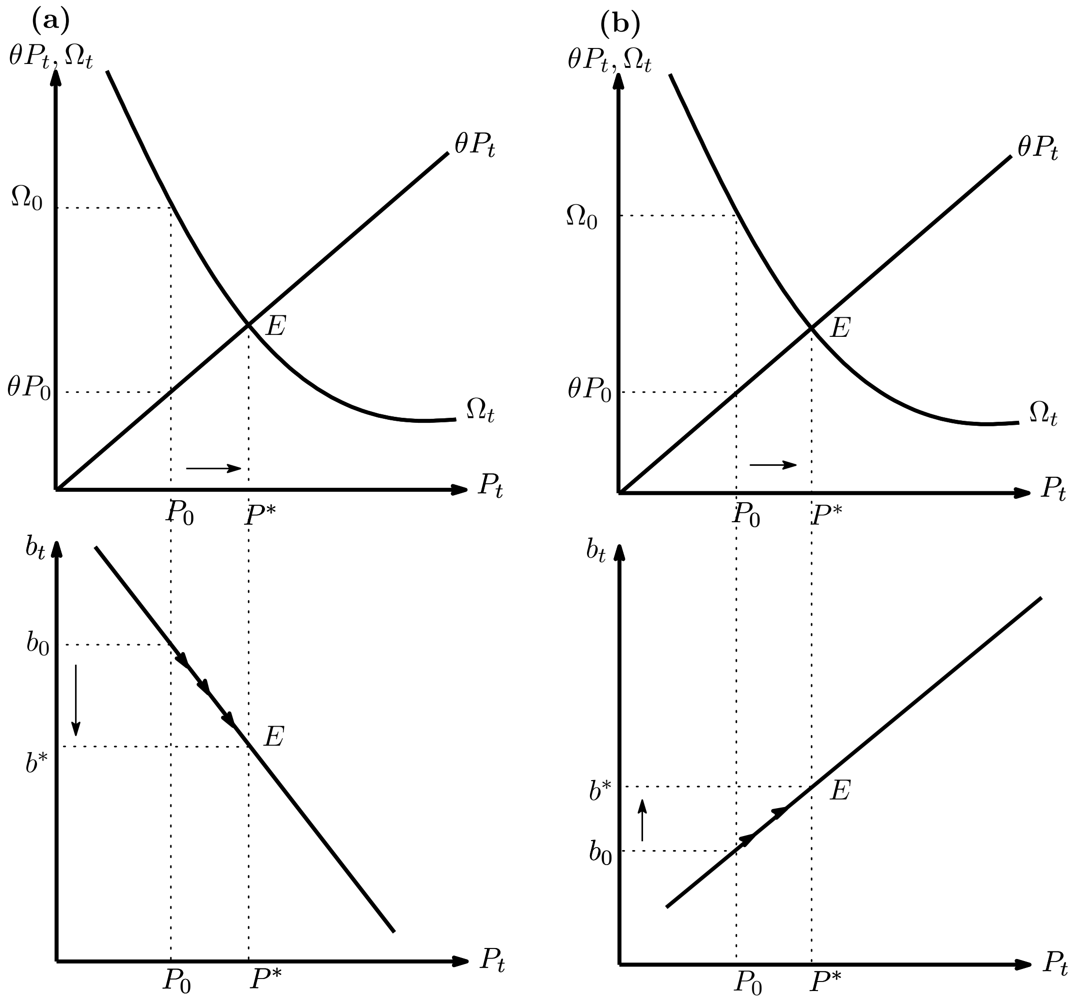


Figure 2: Effects of a permanent increase in taxes

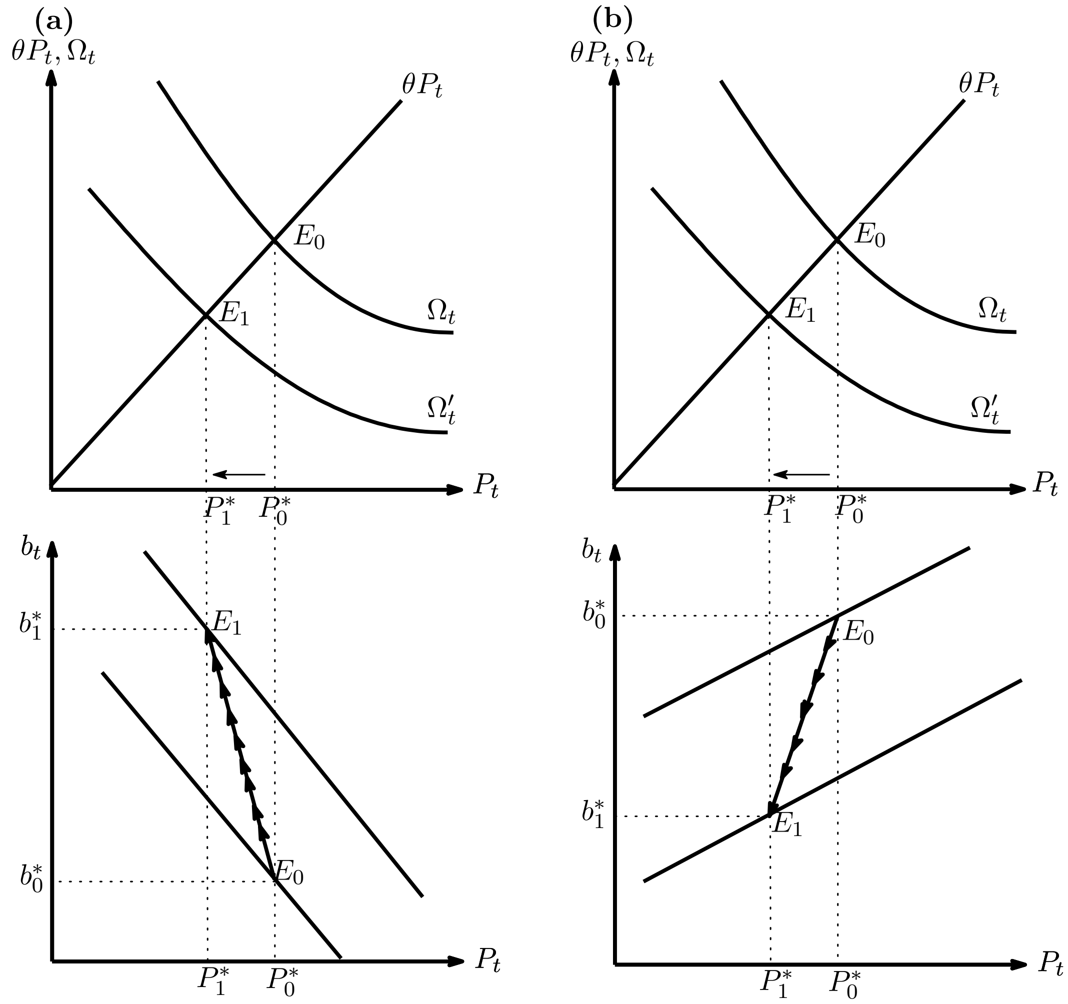


Figure 3: Effects of a temporary increase in taxes

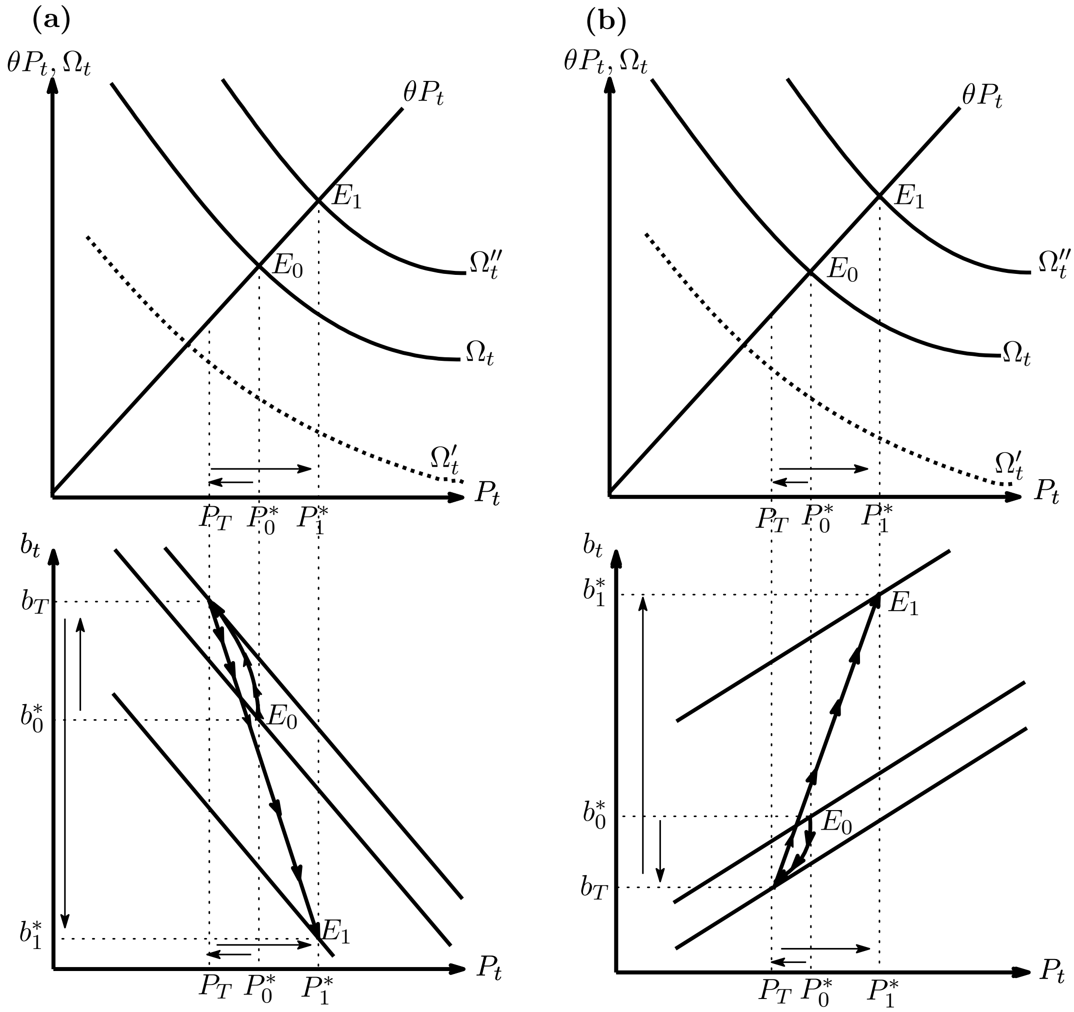


Figure 4: Phase diagram for the optimal path of consumption tax

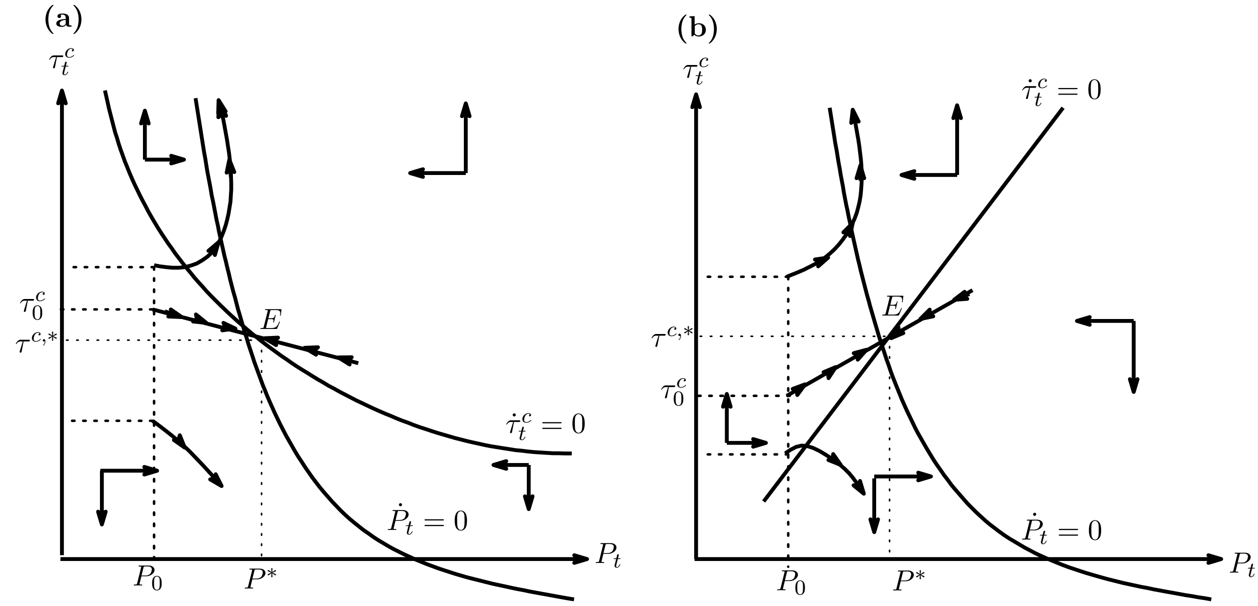
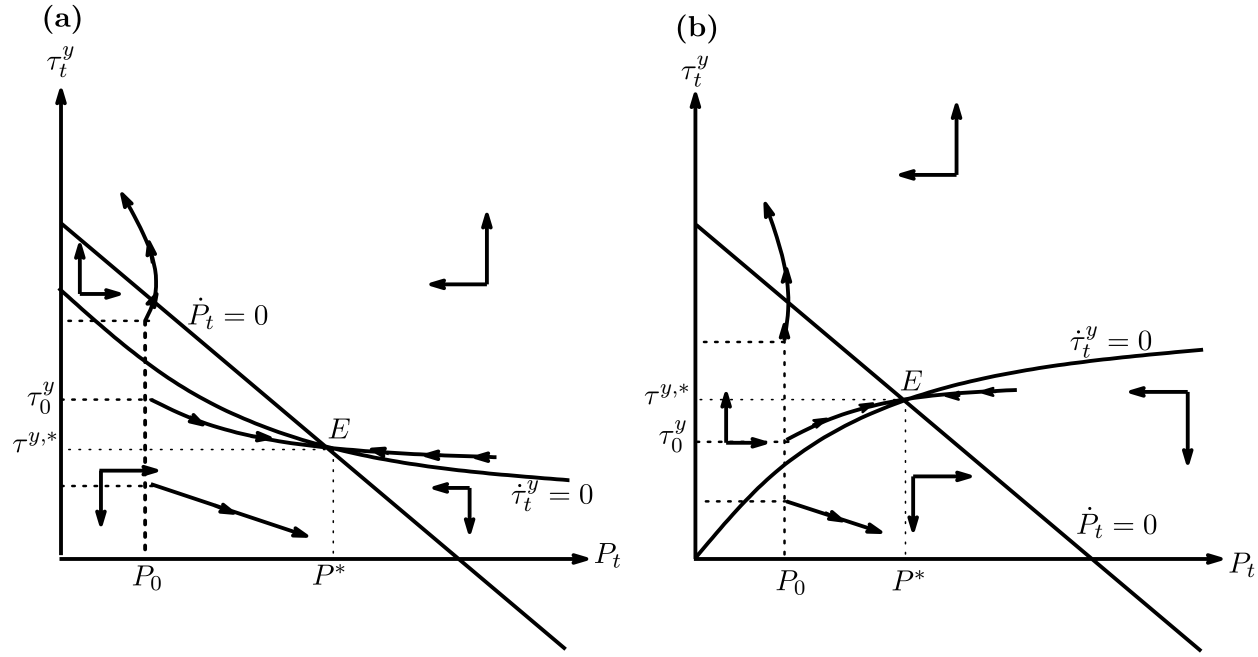


Figure 5: Phase diagram for the optimal path of income tax



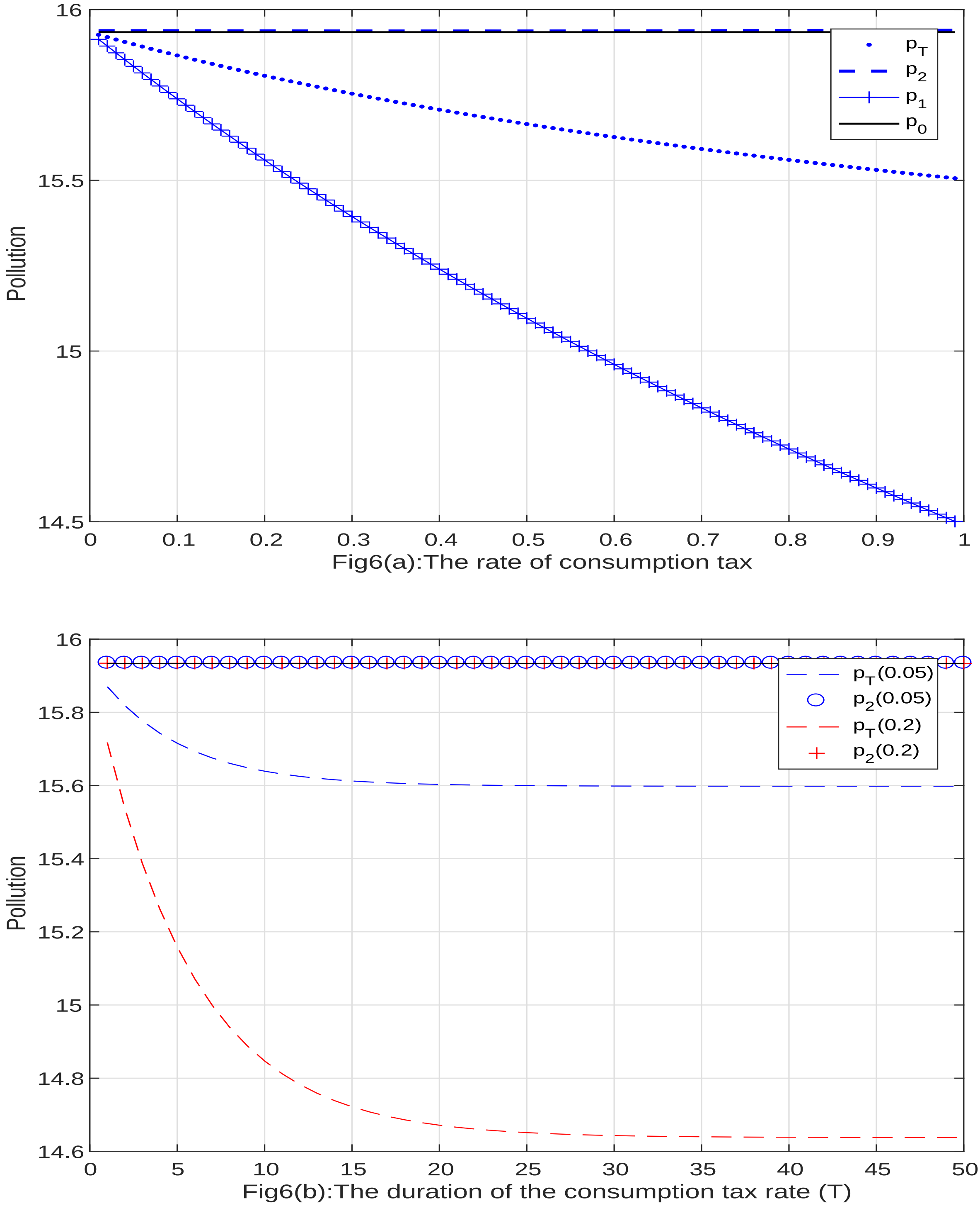


Figure 6: Effects of a temporary change in consumption tax rate

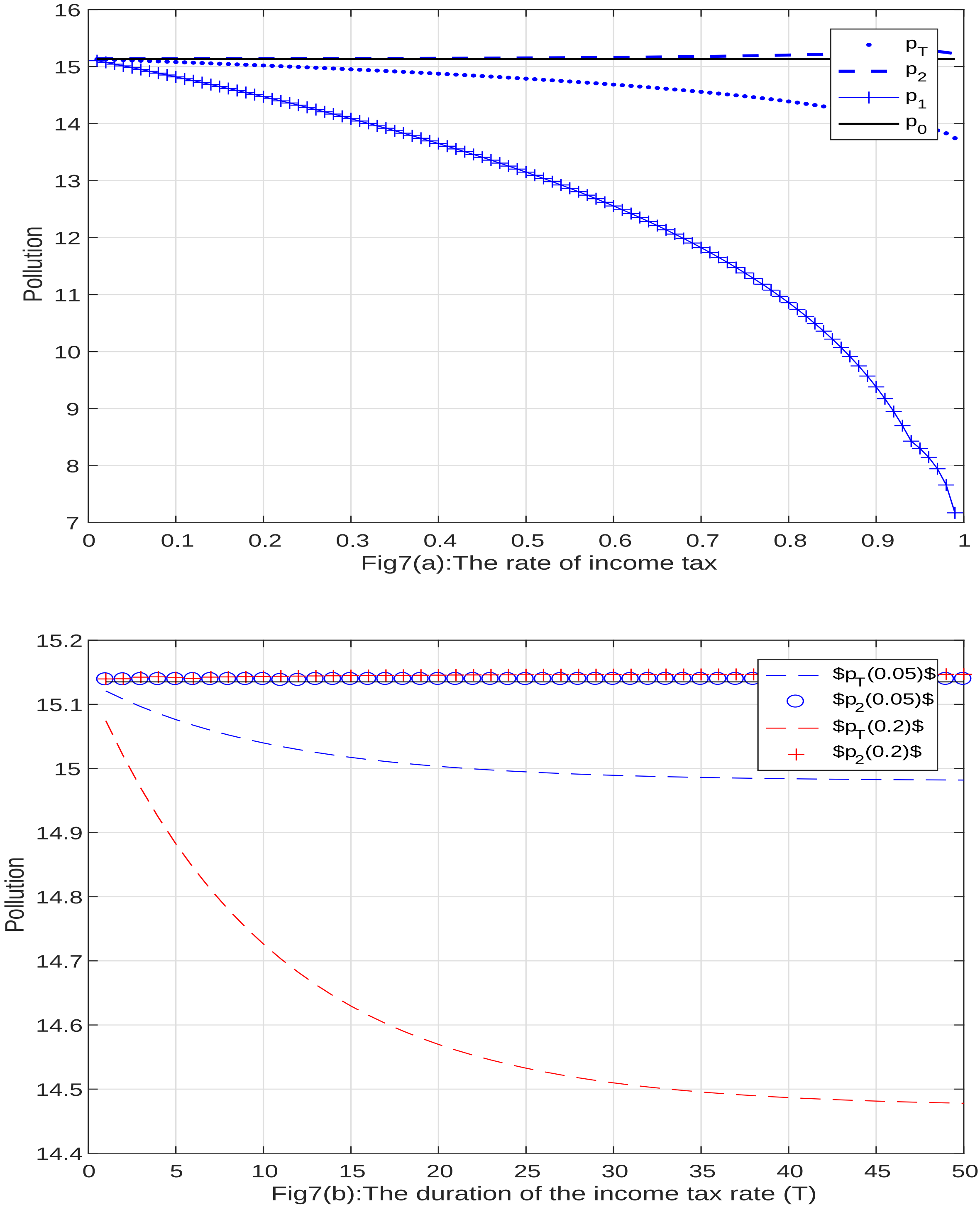


Figure 7: Effects of a temporary change in income tax rate

Appendix A

From (18), the steady-state pollution stock for the given pair of tax rates (τ^c, τ^y) can be written as

$$P^* = P(\bar{\lambda}, \tau^c, \tau^y), \tag{A.1}$$

with the following partial derivatives:

$$P_{\bar{\lambda}} \equiv \frac{\partial P^*}{\partial \bar{\lambda}} = \frac{\Lambda}{\theta_p}, \quad P_{\tau^c} \equiv \frac{\partial P^*}{\partial \tau^c} = \frac{\alpha_c G'(c^*)}{\theta_p}\frac{\partial c^*}{\partial \tau^c}(< 0), \quad P_{\tau^y} \equiv \frac{\partial P^*}{\partial \tau^y} = \frac{\alpha_y N'(l^* f(\bar{k}))f(\bar{k})}{\theta_p}\frac{\partial l^*}{\partial \tau^y}(< 0),$$

where

$$\theta_p \equiv \theta - \alpha_c G'(c^*)\frac{\partial c^*}{\partial P^*} - \alpha_y N'(l^* f(\bar{k}))f(\bar{k})\frac{\partial l^*}{\partial P^*}(> 0),$$
$$\Lambda \equiv \alpha_c G'(c^*)\frac{\partial c^*}{\partial \bar{\lambda}} + \alpha_y N'(l^* f(\bar{k}))f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} < (>) 0 \quad \text{if } \alpha_c > 0 \text{ and } \alpha_y = 0 \ (\alpha_c = 0 \text{ and } \alpha_y > 0).$$

Substituting (A.1) into (22), we observe that

$$rb^* + f(\bar{k})l\left(P(\bar{\lambda}, \tau^c, \tau^y), \bar{\lambda}, \tau^y\right) = c\left(P(\bar{\lambda}, \tau^c, \tau^y), \bar{\lambda}, \tau^c\right), \tag{A.2}$$

which yields the steady-state level of foreign assets as

$$b^* = B\left(\bar{\lambda}, \tau^c, \tau^y\right), \tag{A.3}$$

whose partial derivatives are

$$\begin{aligned}
B_{\bar{\lambda}} &\equiv \frac{\partial b^*}{\partial \bar{\lambda}} = \frac{1}{r}\left[\frac{\partial c^*}{\partial P^*}P_{\bar{\lambda}} + \frac{\partial c^*}{\partial \bar{\lambda}} - f(\bar{k})\left(\frac{\partial l^*}{\partial P^*}P_{\bar{\lambda}} + \frac{\partial l^*}{\partial \bar{\lambda}}\right)\right], \\
B_{\tau^c} &\equiv \frac{\partial b^*}{\partial \tau^c} = \frac{1}{r}\left(\frac{\partial c^*}{\partial P^*}P_{\tau^c} + \frac{\partial c^*}{\partial \tau^c} - f(\bar{k})\frac{\partial l^*}{\partial P^*}P_{\tau^c}\right) \\
&= \underbrace{\frac{1}{r\theta_p}\frac{\partial c^*}{\partial \tau^c}}_{(-)}\underbrace{\left[\theta - f(\bar{k})\frac{\partial l^*}{\partial P^*}\left(\alpha_c G'(c^*) + \alpha_y N'(l^* f(\bar{k}))\right)\right]}_{(+)}(< 0), \\
B_{\tau^y} &\equiv \frac{\partial b^*}{\partial \tau^y} = \frac{1}{r}\left[\frac{\partial c^*}{\partial P^*}P_{\tau^y} - f(\bar{k})\left(\frac{\partial l^*}{\partial \tau^y} + \frac{\partial l^*}{\partial P^*}P_{\tau^y}\right)\right], \\
&= \underbrace{-\frac{f(\bar{k})}{r\theta_p}\frac{\partial l^*}{\partial \tau^y}}_{(+)}\underbrace{\left[\theta - \frac{\partial c^*}{\partial P^*}(\alpha_c G'(c^*) + \alpha_y N'(l^* f(\bar{k})))\right]}_{(+)}(> 0).
\end{aligned}$$

Substituting (A.1) and (A.3) into (24), we obtain

$$b_0 - B(\bar{\lambda}, \tau^c, \tau^y) = \frac{P(\bar{\lambda}, \tau^c, \tau^y) - P_0}{\rho - \mu}\left(f(\bar{k})\frac{\partial l^*}{\partial P^*} - \frac{\partial c^*}{\partial P^*}\right). \tag{A.4}$$

Totally differentiating (A.4), it follows that

$$D_1 d\bar{\lambda} = D_2 d\tau^c + D_3 d\tau^y, \tag{A.5}$$

where

$$D_1 = -B_{\bar{\lambda}} - \frac{1}{\rho - \mu} P_{\bar{\lambda}} \left(f(\bar{k}) \frac{\partial l^*}{\partial P^*} - \frac{\partial c^*}{\partial P^*} \right),$$
$$D_2 = B_{\tau^c} + \frac{1}{\rho - \mu} P_{\tau^c} \left(f(\bar{k}) \frac{\partial l^*}{\partial P^*} - \frac{\partial c^*}{\partial P^*} \right),$$
$$D_3 = B_{\tau^y} + \frac{1}{\rho - \mu} P_{\tau^y} \left(f(\bar{k}) \frac{\partial l^*}{\partial P^*} - \frac{\partial c^*}{\partial P^*} \right).$$

With $r = \rho$, D_1 can be calculated as follows:

$$D_1 = \underbrace{\frac{f(\bar{k})}{\rho} \left(\frac{\partial l^*}{\partial \bar{\lambda}_j} - \frac{\partial l^*}{\partial P^*} P_{\bar{\lambda}} \frac{\mu}{\rho - \mu} \right)}_{(\#A1)} + \underbrace{\frac{1}{\rho} \left(\frac{\partial c^*}{\partial P^*} P_{\bar{\lambda}} \frac{\mu}{\rho - \mu} - \frac{\partial c^*}{\partial \bar{\lambda}} \right)}_{(\#A2)}.$$

The sign of $(\#A1)$ is shown to be positive:

$$(\#A1) = \underbrace{\frac{f}{\rho \theta_p}}_{(+)} \left\{ \underbrace{\frac{\partial l^*}{\partial \bar{\lambda}}}_{(+)} \theta \underbrace{-\alpha_c G'}_{(-)} \underbrace{\left(\frac{\partial l^*}{\partial \bar{\lambda}} \frac{\partial c^*}{\partial P^*} + \frac{\mu}{\rho - \mu} \frac{\partial l^*}{\partial P_j^*} \frac{\partial c^*}{\partial \bar{\lambda}} \right)}_{(-)} \underbrace{-\alpha_y N' f \frac{\partial l^*}{\partial \bar{\lambda}} \frac{\partial l^*}{\partial P^*}}_{(+)} \underbrace{\left(1 + \frac{\mu}{\rho - \mu} \right)}_{(+)} \right\} > 0,$$

where we use $0 < \frac{-\mu}{\rho - \mu} < 1$. We also find that $(\#A2)$ is positive:

$$(\#A2) = \underbrace{-\frac{1}{\rho \theta_p}}_{(-)} \left\{ \underbrace{\theta \frac{\partial c^*}{\partial \bar{\lambda}}}_{(-)} \underbrace{-\alpha_c G' \frac{\partial c^*}{\partial \bar{\lambda}} \frac{\partial c^*}{\partial P^*}}_{(-)} \underbrace{\left(1 + \frac{\mu}{\rho - \mu} \right)}_{(+)} \underbrace{-\alpha_y N' f}_{(-)} \underbrace{\left(\frac{\partial c^*}{\partial \bar{\lambda}} \frac{\partial l^*}{\partial P^*} + \frac{\mu}{\rho - \mu} \frac{\partial c^*}{\partial P^*} \frac{\partial l^*}{\partial \bar{\lambda}} \right)}_{(+)} \right\} > 0.$$

Therefore, it follows that $D_1 > 0$.

We next consider D_2 in (A.5), which can be rewritten as

$$D_2 = \underbrace{\frac{1}{\rho} \left[\frac{\partial c^*}{\partial \tau^c} + \frac{\partial c^*}{\partial P^*} P_{\tau^c} \left(1 - \frac{\rho}{\rho - \mu} \right) \right]}_{(\#A3)} \underbrace{- \frac{f(\bar{k})}{\rho} \frac{\partial l^*}{\partial P^*} P_{\tau^c} \left(1 - \frac{\rho}{\rho - \mu} \right)}_{(-)},$$

where $(\#A3)$ is negative:

$$(\#A3) = \frac{1}{\rho \theta_p} \left[\frac{\partial c^*}{\partial \tau^c} \left(\theta - \alpha_y N' f \frac{\partial l^*}{\partial P^*} \right) - \frac{\rho}{\rho - \mu} \alpha_c G' \frac{\partial c^*}{\partial P^*} \frac{\partial c^*}{\partial \tau^c} \right] < 0.$$

Therefore, we have $D_2 < 0$.

Finally, we consider D_3:

$$D_3 = \underbrace{\frac{1}{\rho} \frac{\partial c^*}{\partial P^*} P_{\tau^y} \left(1 - \frac{\rho}{\rho - \mu} \right)}_{(+)} \underbrace{- \frac{f(\bar{k})}{\rho} \left[\frac{\partial l^*}{\partial \tau^y} + \frac{\partial l^*}{\partial P^*} P_{\tau^y} \left(1 - \frac{\rho}{\rho - \mu} \right) \right]}_{(\#A4)}.$$

Since

$$(\#A4) = -\frac{f}{\rho \theta_p} \left[\frac{\partial l^*}{\partial \tau^y} \left(\theta - \alpha_c G' \frac{\partial c^*}{\partial P^*} \right) - \frac{\rho}{\rho - \mu} \alpha_y N' f \frac{\partial l^*}{\partial P^*} \frac{\partial l^*}{\partial \tau^y} \right] > 0,$$

D_3 is positive.

Therefore, from (A.5), we can obtain the following:

$$\bar{\lambda} = L(\tau^c, \tau^y), \tag{A.6}$$

where

$$L_{\tau^c} \equiv \frac{\partial \bar{\lambda}}{\partial \tau^c} = \frac{D_2}{D_1} < 0, \quad L_{\tau^y} \equiv \frac{\partial \bar{\lambda}}{\partial \tau^y} = \frac{D_3}{D_1} > 0.$$

Appendix B

This appendix examines the effects of a permanent increase in each tax rate on the long-run level of pollution stock $P_0^* = P(\bar{\lambda}_0, \tau_0^c, \tau_0^y) = P(L(\tau_0^c, \tau_0^y), \tau_0^c, \tau_0^y)$ to prove Proposition 2.

We begin with the case in which the consumption tax rate increases permanently. From (A.1) and (A.6), we observe that

$$
\begin{aligned}
\frac{dP_0^*}{d\tau_0^c} &= P_{\bar{\lambda}} L_{\tau^c} + P_{\tau^c} \\
&= \frac{1}{D_1}\left(P_{\bar{\lambda}} D_2 + P_{\tau^c} D_1\right) \\
&= \frac{1}{D_1}\left\{\frac{P_{\bar{\lambda}}}{\rho}\left[\frac{\partial c^*}{\partial \tau^c} + \left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right) P_{\tau^c}\left(1 - \frac{\rho}{\rho-\mu}\right)\right]\right. \\
&\qquad \left. + \frac{P_{\tau^c}}{\rho}\left[f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} - \frac{\partial c^*}{\partial \bar{\lambda}} + P_{\bar{\lambda}}\left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right)\frac{\mu}{\rho-\mu}\right]\right\} \\
&= \frac{1}{D_1 \rho}\left[P_{\bar{\lambda}}\frac{\partial c^*}{\partial \tau^c} + P_{\tau^c}\left(f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} - \frac{\partial c^*}{\partial \bar{\lambda}}\right)\right] \\
&= \frac{1}{D_1 \theta_p \rho}\left[\left(\alpha_c G' \frac{\partial c^*}{\partial \bar{\lambda}} + \alpha_y N' f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right)\frac{\partial c^*}{\partial \tau^c} + \alpha_c G' \frac{\partial c^*}{\partial \tau^c}\left(f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} - \frac{\partial c^*}{\partial \bar{\lambda}}\right)\right] \\
&= \underbrace{\frac{\alpha_c G' + \alpha_y N'}{D_1 \theta_p \rho}}_{(+)} f(\bar{k}) \underbrace{\frac{\partial l^*}{\partial \bar{\lambda}}}_{(+)} \underbrace{\frac{\partial c^*}{\partial \tau^c}}_{(-)} < 0.
\end{aligned}
\tag{B.1}
$$

Therefore, a permanent increase in τ^c unambiguously reduces the steady-state pollution stock.

As for the effect of an increase in the income tax rate, from (A.1) and (A.6), we observe that

$$
\begin{aligned}
\frac{dP_0^*}{d\tau_0^y} &= P_{\bar{\lambda}} L_{\tau^y} + P_{\tau^y} \\
&= \frac{1}{D_1}\left(P_{\bar{\lambda}} D_3 + P_{\tau^y} D_1\right) \\
&= \frac{1}{D_1}\left\{\frac{P_{\bar{\lambda}}}{\rho}\left[-f(\bar{k})\frac{\partial l^*}{\partial \tau^y} + \left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right) P_{\tau^y}\left(1 - \frac{\rho}{\rho-\mu}\right)\right]\right. \\
&\qquad \left. + \frac{P_{\tau^y}}{\rho}\left[f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} - \frac{\partial c^*}{\partial \bar{\lambda}} + P_{\bar{\lambda}}\left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right)\frac{\mu}{\rho-\mu}\right]\right\} \\
&= \frac{1}{D_1 \rho}\left[-P_{\bar{\lambda}} f(\bar{k})\frac{\partial l^*}{\partial \tau^y} + P_{\tau^y}\left(f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} - \frac{\partial c^*}{\partial \bar{\lambda}}\right)\right] \\
&= \frac{1}{D_1 \theta_p \rho}\left[-\left(\alpha_c G' \frac{\partial c^*}{\partial \bar{\lambda}} + \alpha_y N' f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right) f(\bar{k})\frac{\partial l^*}{\partial \tau^y} + \alpha_y N' f(\bar{k})\frac{\partial l^*}{\partial \tau^y}\left(f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} - \frac{\partial c^*}{\partial \bar{\lambda}}\right)\right] \\
&= \underbrace{-\frac{\alpha_c G' + \alpha_y N'}{D_1 \theta_p \rho}}_{(-)} f(\bar{k}) \underbrace{\frac{\partial l^*}{\partial \tau^y}}_{(-)} \underbrace{\frac{\partial c^*}{\partial \bar{\lambda}}}_{(-)} < 0.
\end{aligned}
\tag{B.2}
$$

Therefore, a permanent increase in τ^y unambiguously reduces the steady-state pollution stock.

The signs of (26a) **and** (26b) Eq.(26a) can be rewritten as follows:

$$
\begin{aligned}
&\alpha_c G'(c^*)\frac{\partial c^*}{\partial \tau^c} + \left(\alpha_c G'(c^*)\frac{\partial c^*}{\partial \bar{\lambda}} + \alpha_y N'(l^* f(\bar{k}))f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right) L_{\tau^c} \\
&= \frac{\alpha_c G'}{D_1}\left[D_1 \frac{\partial c^*}{\partial \tau^c} + \left(\frac{\partial c^*}{\partial \bar{\lambda}} + \frac{\alpha_y N'}{\alpha_c G'} f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right) D_2\right] \\
&= \frac{\alpha_c G'}{D_1 \rho}\left\{\left[f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} - \frac{\partial c^*}{\partial \bar{\lambda}} + P_{\bar{\lambda}}\left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right)\frac{\mu}{\rho-\mu}\right]\frac{\partial c^*}{\partial \tau^c}\right. \\
&\qquad \left. + \left(\frac{\partial c^*}{\partial \bar{\lambda}} + \frac{\alpha_y N'}{\alpha_c G'} f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right)\left[\frac{\partial c^*}{\partial \tau^c} + \left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right) P_{\tau^c}\left(1 - \frac{\rho}{\rho-\mu}\right)\right]\right\} \\
&= \frac{\alpha_c G'}{D_1 \rho}\left\{\left(f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} + \frac{\alpha_y N'}{\alpha_c G'} f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right)\frac{\partial c^*}{\partial \tau^c}\right. \\
&\qquad \left. + \left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right)\frac{\mu}{\rho-\mu}\left[P_{\bar{\lambda}}\frac{\partial c^*}{\partial \tau^c} - \left(\frac{\partial c^*}{\partial \bar{\lambda}} + \frac{\alpha_y N'}{\alpha_c G'} f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right) P_{\tau^c}\right]\right\} \\
&= \frac{\alpha_c G'}{D_1 \rho}\left(1 + \frac{\alpha_y N'}{\alpha_c G'}\right) f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\frac{\partial c^*}{\partial \tau^c} < 0.
\end{aligned}
\tag{B.3}
$$

Analogously, (26b) can be rewritten as follows:

$$
\begin{aligned}
&\alpha_y N'(l^* f(\bar{k}))f(\bar{k})\frac{\partial l^*}{\partial \tau^y} + \left(\alpha_c G'(c^*)\frac{\partial c^*}{\partial \bar{\lambda}} + \alpha_y N'(l^* f(\bar{k}))f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right) L_{\tau^y} \\
&= \frac{\alpha_y N'}{D_1}\left[D_1 f(\bar{k})\frac{\partial l^*}{\partial \tau^y} + \left(\frac{\alpha_c G'}{\alpha_y N'}\frac{\partial c^*}{\partial \bar{\lambda}} + f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right) D_3\right] \\
&= \frac{\alpha_y N'}{D_1 \rho}\left\{\left[f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}} - \frac{\partial c^*}{\partial \bar{\lambda}} + P_{\bar{\lambda}}\left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right)\frac{\mu}{\rho-\mu}\right] f(\bar{k})\frac{\partial l^*}{\partial \tau^y}\right. \\
&\qquad \left. + \left(\frac{\alpha_c G'}{\alpha_y N'}\frac{\partial c^*}{\partial \bar{\lambda}} + f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right)\left[-f(\bar{k})\frac{\partial l^*}{\partial \tau^y} + \left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right) P_{\tau^y}\left(1 - \frac{\rho}{\rho-\mu}\right)\right]\right\} \\
&= \frac{\alpha_y N'}{D_1 \rho}\left\{-\left(\frac{\partial c^*}{\partial \bar{\lambda}} + \frac{\alpha_c G'}{\alpha_y N'}\frac{\partial c^*}{\partial \bar{\lambda}}\right) f(\bar{k})\frac{\partial l^*}{\partial \tau^y}\right. \\
&\qquad \left. + \left(\frac{\partial c^*}{\partial P^*} - f(\bar{k})\frac{\partial l^*}{\partial P^*}\right)\frac{\mu}{\rho-\mu}\left[P_{\bar{\lambda}} f(\bar{k})\frac{\partial l^*}{\partial \tau^y} - \left(\frac{\alpha_c G'}{\alpha_y N'}\frac{\partial c^*}{\partial \bar{\lambda}} + f(\bar{k})\frac{\partial l^*}{\partial \bar{\lambda}}\right) P_{\tau^y}\right]\right\} \\
&= -\frac{\alpha_y N'}{D_1 \rho}\left(1 + \frac{\alpha_c G'}{\alpha_y N'}\right)\frac{\partial c^*}{\partial \bar{\lambda}} f(\bar{k})\frac{\partial l^*}{\partial \tau^y} < 0.
\end{aligned}
\tag{B.4}
$$

Appendix C

In this appendix, we examine the effects of a temporary change in the tax rates on the long-run pollution stock to prove Proposition 3. During the period between time 0 and T, the government sets tax rate τ_1^i, which differs from the original rate, τ_0^i, for $i = c, y$. From time T onward, the government reverts to the original level, τ_0^i. In the following, we call the former period (i.e., $0 \le t < T$) Period 1 and the latter period (i.e., $t \ge T$) Period 2.

Period 1: $0 \le t < T$

During Period 1, in which the tax rates are τ_1^c and τ_1^y, the economy moves along an unstable transitional path:

$$
P_t = P_1^* + R_1 e^{\mu_1^s t} + R_2 e^{\mu_1^u t}, \tag{C.1a}
$$

$$b_t = b_1^* + \left(f(\bar{k}) \frac{\partial l_1^*}{\partial P_1^*} - \frac{\partial c_1^*}{\partial P_1^*} \right) \left(\frac{R_1}{\mu_1^s - r} e^{\mu_1^s t} + \frac{R_2}{\mu_1^u - r} e^{\mu_1^u t} \right), \tag{C.1b}$$

where μ_1^s and μ_1^u represent stable and unstable roots, respectively, under $(\tau_1^c, \tau_1^y, P_0, b_0)$, and P_1^*, b_1^* and $\bar{\lambda}^*$ are steady-state equilibrium values when the pair of taxes is given by (τ_1^c, τ_1^y).

Given the initial levels of the pollution stock and foreign assets, we can determine arbitrary constants R_1 and R_2. Setting $t = 0$ in (C.1a) and (C.1b) and calculating them, we obtain the following:

$$R_1 = \frac{(\mu_1^u - r)(\mu_1^s - r)}{\mu_1^s - \mu_1^u} \left(\frac{P_0 - P_1^*}{\mu_1^u - r} - \frac{b_0 - b_1^*}{f(\bar{k}) \frac{\partial l_1^*}{\partial P_1^*} - \frac{\partial c_1^*}{\partial P_1^*}} \right), \tag{C.2a}$$

$$R_2 = \frac{(\mu_1^u - r)(\mu_1^s - r)}{\mu_1^u - \mu_1^s} \left(\frac{P_0 - P_1^*}{\mu_1^s - r} - \frac{b_0 - b_1^*}{f(\bar{k}) \frac{\partial l_1^*}{\partial P_1^*} - \frac{\partial c_1^*}{\partial P_1^*}} \right). \tag{C.2b}$$

Given the initial levels of foreign assets and pollution, $(b_0,\ P_0)$, the steady-state levels, expressed by P_1^* and b_1^*, are determined under the new tax rate, τ_1^i ($i = c$ or y), where we use (A.6):

$$P_1^* = P(\bar{\lambda}_1, \tau_1^c, \tau_1^y) = P(L(\tau_1^c, \tau_1^y, b_0, p_0), \tau_1^c, \tau_1^y), \tag{C.3a}$$

$$b_1^* = B(\bar{\lambda}_1, \tau_1^c, \tau_1^y) = B(L(\tau_1^c, \tau_1^y, b_0, p_0), \tau_1^c, \tau_1^y), \tag{C.3b}$$

$$\bar{\lambda}_1 = L(\tau_1^c, \tau_1^y, b_0, p_0). \tag{C.3c}$$

Over this period, it must hold that $\bar{\lambda}_0 \neq \bar{\lambda}_1$, because the shadow value jumps after the initial change in the tax rates.

Period 2: $T \geq t$

During Period 2, in which the tax rates return to τ_0^c and τ_0^y, the economy follows a stable path as follows:

$$P_t = P_2^* + R_1' e^{\mu_2^s t}, \tag{C.4a}$$

$$b_t = b_2^* + \left(f(\bar{k}) \frac{\partial l_2^*}{\partial P_2^*} - \frac{\partial c_2^*}{\partial P_2^*} \right) \frac{R_1' e^{\mu_2^s t}}{\mu_2^s - r}. \tag{C.4b}$$

We first look at the determination of the remaining arbitrary constant R_1'. By using (C.1a) and (C.4a) at time T, we can show that

$$P_2^* + R_1' e^{\mu_2^s T} = P_1^* + R_1 e^{\mu_1^s T} + R_2 e^{\mu_1^u T},$$

where constants R_1 and R_2 are determined by (C.2a) and (C.2b), respectively. Therefore, R_1' is derived as

$$R_1' = e^{-\mu_2^s T} \left(P_1^* - P_2^* + R_1 e^{\mu_1^s T} + R_2 e^{\mu_1^u T} \right).$$

The steady-state levels of P_2^* and b_2^* are determined by

$$P_2^* = P(\bar{\lambda}_2^*, \tau_0^c, \tau_0^y) = P(L(\tau_0^c, \tau_0^y, b_T, p_T), \tau_0^c, \tau_0^y), \tag{C.5a}$$

$$b_2^* = B(\bar{\lambda}_2^*, \tau_0^c, \tau_0^y) = B(L(\tau_0^c, \tau_0^y, b_T, p_T), \tau_0^c, \tau_0^y), \tag{C.5b}$$

$$\bar{\lambda}_2^* = L(\tau_0^c, \tau_0^y, b_T, p_T). \tag{C.5c}$$

Note that the level of the shadow value does not change, $\bar{\lambda}_1 = \bar{\lambda}_2$, because the household anticipates that, under the assumption of perfect foresight, the tax rates are back to their original levels.

We denote the policy changes by $d\tau^i \equiv \tau_1^i - \tau_0^i$, $i = c, y$. Therefore, the changes in the steady-state pollution stocks can be approximated as follows:

$$P_2^* - P_1^* = P(\bar{\lambda}_2^*, \tau_0^c, \tau_0^y) - P(\bar{\lambda}_1^*, \tau_1^c, \tau_1^y) = -P_{\tau^c} d\tau^c - P_{\tau^y} d\tau^y, \tag{C.6a}$$

$$P_1^* - P_0^* = P(\bar{\lambda}_1^*, \tau_1^c, \tau_1^y) - P(\bar{\lambda}_0^*, \tau_0^c, \tau_0^y) = P_{\bar{\lambda}} \left(L_{\tau^c} d\tau^c + L_{\tau^y} d\tau^y \right) + P_{\tau^c} d\tau^c + P_{\tau^y} d\tau^y. \tag{C.6b}$$

Note that equality $\bar{\lambda}_1 = \bar{\lambda}_2$ holds.

From (C.6a) and (C.6b), the effects of the temporary change in each tax rate on pollution can be derived as follows:

$$P_2^* - P_0^* = P_{\bar{\lambda}} \left(L_{\tau^c} d\tau^c + L_{\tau^y} d\tau^y \right), \tag{C.7}$$

where $P_{\bar{\lambda}}$ is given by (A.1) and L_{τ^c} and L_{τ^y} by (A.6). This completes the proof of Proposition 3.

Appendix D

To prove Proposition 4, we begin with the conditions under which the steady-state equilibrium is uniquely determined. In the following, we denote $\tilde{c}^* = \tilde{c}(\tilde{P}^*, \bar{\bar{\lambda}}^*, \tilde{\phi}^*)$ and $\tilde{l}^* = \tilde{l}(\tilde{P}^*, \bar{\bar{\lambda}}^*, \tilde{\phi}^*)$.

From (32a), given the shadow value $\bar{\bar{\lambda}}^*$, the steady-state pollution stock can be expressed as

$$\tilde{P}^* = \tilde{P}(\tilde{\phi}^*), \tag{D.1}$$

which has the following derivative:

$$\frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*} = \frac{\alpha_c G'(\tilde{c}^*) \frac{\partial \tilde{c}^*}{\partial \tilde{\phi}^*} + \alpha_y N'(\tilde{l}^* f(\bar{k})) \frac{\partial \tilde{l}^*}{\partial \tilde{\phi}^*}}{\theta - \alpha_c G'(\tilde{c}^*) \frac{\partial \tilde{c}^*}{\partial \tilde{P}^*} - \alpha_y N'(\tilde{l}^* f(\bar{k})) f(\bar{k}) \frac{\partial \tilde{l}^*}{\partial \tilde{P}^*}} (< 0).$$

Let us consider the case in which $\tilde{\phi}^* \to 0$. Then, from (29a) and (29b), we observe that

$$u_c(\tilde{c}^*, \tilde{P}^*) = \bar{\bar{\lambda}}^*, \tag{D.2}$$

$$\omega_l(\tilde{l}^*, \tilde{P}^*) = \bar{\bar{\lambda}}^* f(\bar{k}). \tag{D.3}$$

Given the constant shadow value $\bar{\bar{\lambda}}^*$, the marginal utility of consumption and marginal disutility of labor supply are also constant. This means that the pollution stock has a finite value of

$\tilde{P}^* \to \tilde{\tilde{P}}^*$ as $\tilde{\phi}^* \to 0$ and that $\tilde{c}^* \to \tilde{\tilde{c}}^*$ and $\tilde{l}^* \to \tilde{\tilde{l}}^*$ as $\tilde{\phi}^* \to 0$, where the levels of consumption, labor supply, and pollution stock ($\tilde{\tilde{P}}^*$) must satisfy equations (D.2), (D.3), and (32a).[19]

We next consider the case in which $\tilde{\phi}^* \to \infty$. Then, equations (29a) and (29b) can be rewritten as

$$u_c(\tilde{c}^*, \tilde{P}^*) = \infty, \tag{D.4}$$

$$\omega_l(\tilde{l}^*, \tilde{P}^*) = 0, \tag{D.5}$$

where we assume that $\bar{\lambda} = \infty$ in this case, so that the value of the left-hand side of (D.5) is always non-negative. Considering (D.4), we observe that the level of the marginal utility of consumption is infinite so that either consumption or pollution stock must be 0. Furthermore, from (D.5), we can similarly observe that the level of the marginal disutility of labor supply is 0 so that either the level of labor supply or that of the pollution stock must be 0. Then, we find that all levels of consumption, labor supply, and pollution stock must be 0, to satisfy (32a). As a result, it holds that $\tilde{P}^* \to 0$ as $\tilde{\phi}^* \to \infty$.

We now substitute (D.1) into (32b) as follows:

$$\Gamma(\tilde{\phi}^*) \equiv (\rho+\theta)\tilde{\phi}^* - \Big\{\omega_P(\tilde{l}(\tilde{P}(\tilde{\phi}^*), \bar{\tilde{\lambda}}^*, \tilde{\phi}^*), \tilde{P}(\tilde{\phi}^*)) - u_P(\tilde{c}(\tilde{P}(\tilde{\phi}^*), \bar{\tilde{\lambda}}^*, \tilde{\phi}^*), \tilde{P}(\tilde{\phi}^*))\Big\} = 0. \tag{D.6}$$

First, we observe that

$$\lim_{\tilde{\phi}^* \to 0} \Gamma(\tilde{\phi}^*) = -\Big\{\omega_P(\tilde{\tilde{l}}, \tilde{\tilde{P}}) - u_P(\tilde{\tilde{c}}, \tilde{\tilde{P}})\Big\} < 0, \tag{D.7}$$

$$\lim_{\tilde{\phi}^* \to \infty} \Gamma(\tilde{\phi}^*) = \infty - \Big\{\underbrace{\omega_P(0,0)}_{=0} - \underbrace{u_P(0,0)}_{=0/0}\Big\} > 0, \tag{D.8}$$

where we note that $u_P(\tilde{\tilde{c}}, \tilde{\tilde{P}}) < 0$ from (5a).

Finally, we differentiate (D.6) with respect to the shadow value of the pollution stock, $\tilde{\phi}^*$, as follows:

$$\begin{aligned}\Gamma'(\tilde{\phi}^*) = \rho + \theta - \Bigg[\bigg(\omega_{lP}(\tilde{l}^*, \tilde{P}^*)\frac{\partial \tilde{l}^*}{\partial \tilde{P}^*} + \omega_{PP}(\tilde{l}^*, \tilde{P}^*)\bigg)\frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*} + \omega_{lP}(\tilde{l}^*, \tilde{P}^*)\frac{\partial \tilde{l}^*}{\partial \tilde{\phi}^*}\Bigg] \\ + \Bigg[\bigg(u_{cP}(\tilde{c}^*, \tilde{P}^*)\frac{\partial \tilde{c}^*}{\partial \tilde{P}} + u_{PP}(\tilde{c}, \tilde{P})\bigg)\frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*} + u_{cP}(\tilde{c}, \tilde{P})\frac{\partial \tilde{c}}{\partial \tilde{\phi}}\Bigg].\end{aligned} \tag{D.9}$$

[19]The pollution stock can be neither 0 nor infinity. To observe this, suppose that $\tilde{P}^* \to 0$ when $\tilde{\phi}^* \to 0$. In this case, from (32a), it must hold that $\tilde{c}^* \to 0$ and $\tilde{l}^* \to 0$. This means that $\lim_{\tilde{c}^* \to 0} u_c = \infty$ and $\lim_{\tilde{l}^* \to 0} \omega_l = 0$ in light of (5c), which contradicts (D.2) and (D.3). Similarly, suppose that $\tilde{P}^* \to \infty$. Then, from (32a), it must hold that $\tilde{c}^* \to \infty$ and/or $\tilde{l}^* \to \infty$, and thus, $\lim_{\tilde{c}^* \to \infty} u_c = \bar{\tilde{\lambda}}^*$ or $\lim_{\tilde{l}^*} \omega_l = \bar{\tilde{\lambda}}^* f(\bar{k})$. However, these equations contradict (5c). As a result, the pollution stock must be finite when $\tilde{\phi}^* \to 0$.

The case in which $\alpha_c > 0$ and $\alpha_y = 0$ Assuming that $\alpha_c > 0 = \alpha_y$, (D.9) can be rewritten as[20]

$$\begin{aligned}
\text{(D.9)} &= \rho + \theta - \left\{ \frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*} \Phi_l - u_{cP}(\tilde{c}^*, \tilde{P}^*) \left(\frac{\partial \tilde{c}^*}{\partial \tilde{P}^*} \frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*} + \frac{\partial \tilde{c}^*}{\partial \tilde{\phi}^*} \right) \right\} \\
&= \rho + \theta - \left\{ \underbrace{\frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*}}_{(-)} \Phi_l + \underbrace{\frac{\theta u_{cP}(\tilde{c}^*, \tilde{P}^*) \frac{\partial \tilde{c}^*}{\partial \tilde{\phi}^*}}{\theta - \alpha_c G'(\tilde{c}^*) \frac{\partial \tilde{c}^*}{\partial \tilde{P}^*}}}_{(-)} \right\} (> 0),
\end{aligned}$$

where we use (34a). Based on (D.7) and (D.8), we can prove that there is a unique steady-state equilibrium.

The saddle-point stability of the steady-state equilibrium can also be obtained by inspecting the determinant in (#2) of (33) and using (34a):

$$(\#2) = \alpha_c G'(\tilde{c}^*) \left[\underbrace{(\rho + \theta) \frac{\partial \tilde{c}^*}{\partial \tilde{P}^*}}_{(-)} + \underbrace{\frac{\partial \tilde{c}^*}{\partial \tilde{\phi}^*}}_{(-)} \Phi_c \right] \underbrace{-\theta \left(\rho + \theta + u_{cP} \frac{\partial \tilde{c}^*}{\partial \tilde{\phi}^*} \right)}_{(-)} < 0.$$

The case in which $\alpha_c = 0$ and $\alpha_y > 0$ We now assume that $\alpha_c = 0 < \alpha_y$, and hence, (D.9) can be rewritten as[21]

$$\begin{aligned}
\text{(D.9)} &= \rho + \theta - \left\{ \frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*} \Phi_c - \omega_{lP}(\tilde{l}^*, \tilde{P}^*) \left(\frac{\partial \tilde{l}^*}{\partial \tilde{P}^*} \frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*} + \frac{\partial \tilde{l}^*}{\partial \tilde{\phi}^*} \right) \right\} \\
&= \rho + \theta - \left\{ \underbrace{\frac{\partial \tilde{P}^*}{\partial \tilde{\phi}^*}}_{(-)} \Phi_c + \underbrace{\frac{\theta \omega_{lP}(l, P) \frac{\partial \tilde{l}^*}{\partial \tilde{\phi}^*}}{\theta - \alpha_y N'(l f(\bar{k})) f(\bar{k}) \frac{\partial \tilde{l}^*}{\partial \tilde{P}^*}}}_{(-)} \right\} > 0,
\end{aligned}$$

where we use (34b). Thus, the steady-state equilibrium is uniquely determined under (D.7) and (D.8).

Finally, we show the saddle-path stability in the steady-state equilibrium, by using (#2) in (33) and (34b):

$$(\#2) = \alpha_y N'(\tilde{l}^* f(\bar{k})) f(\bar{k}) \left[\underbrace{(\rho + \theta) \frac{\partial \tilde{l}^*}{\partial \tilde{P}^*}}_{(-)} + \underbrace{\frac{\partial \tilde{l}^*}{\partial \tilde{\phi}^*}}_{(-)} \Phi_c \right] \underbrace{-\theta \left(\rho + \theta - \omega_{lP} \frac{\partial \tilde{l}^*}{\partial \tilde{\phi}^*} \right)}_{(-)} < 0.$$

Appendix E

The case in which $\alpha_c > 0$ and $\alpha_y = 0$ Given shadow value $\bar{\lambda}$, we totally differentiate (42b) and obtain the following:[22]

$$\tau^{c,*} = \tau^c(P^*), \tag{E.1}$$

[20]Note that $\frac{\partial \tilde{l}^*}{\partial \tilde{\phi}^*} = 0$ under $\alpha_y = 0$.

[21]Note that $\frac{\partial \tilde{c}^*}{\partial \tilde{\phi}^*} = 0$ under $\alpha_c = 0$.

[22]To simplify the notation, we omit the variables in each function.

where

$$\frac{\partial \tau^{c,*}}{\partial P^*} = \frac{\theta - \alpha_c G' \frac{\partial c^*}{\partial P^*}}{\alpha_c G' \frac{\partial c^*}{\partial \tau^{c,*}}} < 0.$$

Next, we consider the case in which the level of the pollution stock is 0. Then, from (42b), the value of the right-hand side is 0, and hence, the level of the pollution flow, $G(c^*)$, is 0 as well. Since the level of consumption is 0, from (5c), we find that $\lim_{c^* \to 0} u_c \to \infty$, thereby showing that the rate of consumption tax is infinite from (11a). In summary, it holds that $\lim_{P^* \to 0} \tau^c(P^*) = \infty$. Similarly, when the level of the pollution stock is infinite, from (42b), the level of the pollution flow is infinite, which means that the level of consumption is infinite. By using (5c), we observe that $\lim_{c^* \to \infty} u_c \to 0$, and hence, the value of the left-hand side in (11a) is 0, which implies that because the rate of consumption tax is -1, $\lim_{P^* \to \infty} \tau^c(P^*) = -1$.

We now substitute (E.1) into (42a) as follows:

$$\Gamma^c(P^*) \equiv (\rho + \theta)\tau^c(P^*)\bar{\lambda}^* - \alpha_c G'(c(P^*, \bar{\lambda}^*, \tau^c(P^*))) \Big\{ \omega_P(l(P^*, \bar{\lambda}^*), P^*) - u_P(c(P^*, \bar{\lambda}^*, \tau^c(P^*)), P^*) \Big\}$$
$$= 0. \tag{E.2}$$

Then, we find that

$$\lim_{P^* \to 0} \Gamma^c(P^*) = \infty - \alpha_c \underbrace{G'(c(0, \bar{\lambda}^*, \infty))}_{=0} \Big\{ \omega_P(l(0, \bar{\lambda}^*), 0) - u_P(c(0, \bar{\lambda}^*, \infty), 0) \Big\} = \infty > 0. \tag{E.3a}$$

When the rate of consumption tax is infinite, we show that the value of the right-hand side of (11a) becomes infinite. Therefore, from (5c), consumption is 0, and hence, $G'(c(0, \bar{\lambda}, \infty)) = 0$ in (E.3a). Furthermore, it holds that

$$\lim_{P^* \to \infty} \Gamma^c(P^*) = -(\rho + \theta)\bar{\lambda}^* \underbrace{- \alpha_c G'(c(\infty, \bar{\lambda}^*, -1)) \Big\{ \omega_P(l(\infty, \bar{\lambda}^*), \infty) - u_P(c(\infty, \bar{\lambda}^*)) \Big\}}_{\leq 0} < 0. \tag{E.3b}$$

Finally, we can express the negative slope of $\Gamma^c(P^*)$ as follows:[23]

$$\frac{\partial \Gamma^c(P^*)}{\partial P^*} = \underbrace{(\theta + \rho)\bar{\lambda}^* \frac{\partial \tau^{c,*}}{\partial P^*}}_{(-)} + \theta \Bigg(\underbrace{u_{cP} - \frac{G''(\omega_P - u_P)}{G'}}_{(-)} \Bigg) - \alpha_c G' \underbrace{\Phi_l}_{(+)} < 0, \tag{E.3c}$$

where we use (34a). As a result, the steady state is uniquely determined from (E.3a)–(E.3c).

The case in which $\alpha_c = 0$ and $\alpha_y > 0$ The proof is the same as above. First, from (48b), we show that

$$\tau^{y,*} = \tau^y(P^*), \tag{E.4}$$

[23] We use the following:

$$\frac{\partial c^*}{\partial P^*} + \frac{\partial c^*}{\partial \tau^{c,*}} \frac{\partial \tau^{c,*}}{\partial P^*} = \frac{\theta}{\alpha_c G'} > 0.$$

where

$$\frac{\partial \tau^{y,*}}{\partial P^*} = \frac{\theta - \alpha_y N' f \frac{\partial l^*}{\partial P^*}}{\alpha_y N' f \frac{\partial l^*}{\partial \tau^y}} < 0.$$

In addition, when the level of the pollution stock approaches 0, from (48b), we observe that the level of the pollution flow, $N\left(l^* f(\bar{k})\right)$ is 0 and, therefore, the level of labor input is 0 as well. As a result, from (5c), we show that $0 = \bar{\lambda}(1-\tau^{y,*})f(\bar{k})$ in (11b). Therefore, we find that $\lim_{P^*\to 0} \tau^y(P^*) = -1$. On the other hand, assuming that the level of the pollution stock goes to infinity, we find that the level of the pollution flow is infinite in (48b), and hence, the level of labor input becomes infinite. Therefore, from (11b), it holds that $\infty = \bar{\lambda}(1-\tau^{y,*})f(\bar{k})$, meaning that the rate of income tax approaches $-\infty$; it holds that $\lim_{P^*\to\infty} \tau^y(P^*) = -\infty$.

Finally, substituting (E.4) into (48a) yields

$$\begin{aligned} \Gamma^y(P^*) &\equiv (\rho+\theta)\tau^y(P^*)\bar{\lambda}^* - \alpha_y N'(l(P^*,\bar{\lambda},\tau^y(P^*))f(\bar{k}))\left\{\omega_P(l(P^*,\bar{\lambda}^*,\tau^y(P^*))) - u_P(c(P^*,\bar{\lambda}^*),P^*)\right\} \\ &= 0. \end{aligned} \tag{E.5}$$

We obtain the following:

$$\lim_{P^*\to 0} \Gamma^y(P^*) = (\rho+\theta)\bar{\lambda}^* - \alpha_y \underbrace{N'(l(0,\bar{\lambda},1)f(\bar{k}))}_{=0}\left\{\omega_P(l(0,\bar{\lambda},1)) - u_P(c(0,\bar{\lambda}),0)\right\} = (\rho+\theta)\bar{\lambda}^* > 0, \tag{E.6a}$$

where we show that, when $\tau^y = -1$, from (11b), the level of labor input is 0 from (5c), and thus, $N' \to 0$. Next, we can show that

$$\lim_{P^*\to\infty} \Gamma^y(P^*) = -\infty \underbrace{-\alpha_y N'(l(\infty,\bar{\lambda}^*,-\infty)f(\bar{k}))\left\{\omega_P(l(\infty,\bar{\lambda}^*,-\infty)) - u_P(c(\infty,\bar{\lambda}^*),\infty)\right\}}_{\leq 0} = -\infty < 0. \tag{E.6b}$$

Finally, the slope of $\Gamma^y(P^*)$ is given by:[24]

$$\frac{\partial \Gamma^y(P^*)}{\partial P^*} = \underbrace{(\rho+\theta)\bar{\lambda}^* \frac{\partial \tau^{y,*}}{\partial P^*}}_{(-)} - \theta\Bigg(\underbrace{\omega_{lP} + \frac{N''(\omega_P - u_P)}{N'}}_{(+)}\Bigg) - \alpha N' f \underbrace{\Phi_c}_{(+)} < 0. \tag{E.6c}$$

From (E.6a)–(E.6c), we show the uniqueness of the steady state.

[24] We use the following:

$$\frac{\partial l^*}{\partial P^*} + \frac{\partial l^*}{\partial \tau^{y,*}}\frac{\partial \tau^{y,*}}{\partial P^*} = \frac{\theta}{\alpha_y N' f} > 0.$$